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Limit Theorem for the Distribution of Eigenvalues of the Operator of Energy

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We prove the central limit theorem for the distribution of eigenvalues of the energy operator of a continuous quantum mechanical system. We consider one-dimensional and multidimensional systems.

KEY WORDS: Limit theorem; eigenvalues.

1. INTRODUCTION

In the present paper we prove that the distribution of eigenvalues of the energy operator of a system of quantum particles in the box $\Lambda \subset R^{\nu}$, interacting through a two-body potential, is subjected to a Gaussian law if $\Lambda \to \infty$, $N \to \infty$, $|\Lambda|^{-1} N \to d$. We consider the Fermi and Bose systems at all values of the density d in the one-dimensional case and at small values of the density in the multidimensional case. The exact formulation of the problem and the results are given in Section 2. The basic tool we used in the proof of limit theorems in the present paper are the reduced density matrices, which were introduced and investigated in the multidimensional case by J. Ginibre[1-3] and in the one-dimensional case by Yu. Suchov.[4-6] One can easily write formally the moments of the distributions which interest us in terms of derivatives of the reduced density matrices with respect to the parameter β (inverse temperature). In Refs. 1–6 it is shown that under some hypotheses the reduced density matrices can be written in terms of Wiener integrals, such that one can go to the limit as $\Lambda \to \infty$ in the corresponding formulas. The parameter β is contained in these Wiener integrals in a standard way. Thus in order to compute the moments of the investigated distribution, we must first get the expressions of the derivatives

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with respect to the parameter β of some Wiener integrals. Section 3 is devoted to this purpose. By using the above-mentioned expressions of the reduced density matrices and the results of Section 3, we obtain the derivatives with respect to the parameter β of the reduced density matrices, and prove the existence of the limit of these derivatives, when the volume tends to infinity. From these results we derive the central limit theorem in the grand ensemble, and then we prove the central limit theorem in the ensemble of N particles by using the well-known method of proving the local limit theorem for the number of particles.[7,8] The one-dimensional case is considered in Section 4, and the multidimensional case in Section 5.

In the case of noninteracting particles the problem was solved in Ref. 9 for the Maxwell–Boltzmann statistics.

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2. FORMULATION OF THE PROBLEM AND RESULTS

We shall consider a system of N quantum particles in a box $\Lambda \subset R^{\nu}$, which is described by the Schroedinger equation

$$-\tfrac{1}{2}\sum \Delta_j \varphi(x_1,..., x_N) + \sum_{i \neq j} V(x_i - x_j)\, \varphi(x_1,..., x_N) = E\varphi(x_1,..., x_N) \tag{2.1}$$

where φ is a function of the variables $x_j \in \Lambda$, Δ_j is the Laplacian with respect to appropriate variables, $V(x)$ is the potential. Let $H_{N,\Lambda,B}$, $H_{N,\Lambda,F}$ denote the self-adjoint operator, defined by equation (2.1) and the null boundary conditions in the space $L^2_+(\Lambda^N)$ of the symmetric functions (Bose statistics), and $L^2_-(\Lambda^N)$ antisymmetric functions (Fermi statistics), respectively. All formulations and results which we give below, are correct for both statistics. Therefore we shall omit the indexes B, F. It is well known that under wide hypotheses the spectrum of the operator $H_{N,\Lambda}$ consists of eigenvalues of finite multiplicity $E_1(N, \Lambda) \leqslant E_2(N, \Lambda) \leqslant \cdots$; moreover, for all $\beta > 0$

$$Q(N, \Lambda, \beta) = \sum_{\kappa} \exp[-\beta E_\kappa(N, \Lambda)] < +\infty \tag{2.2}$$

Fix $\beta > 0$. Then the mean value of the energy $\langle H_{N,\Lambda} \rangle$ is computed by the formula

$$\langle H_{N,\Lambda} \rangle = Q(N, \Lambda, \beta)^{-1} \sum_{\kappa} E_\kappa(N, \Lambda) \exp[-\beta E_\kappa(N, \Lambda)] \tag{2.3}$$

and the probability $P_\kappa(N, \Lambda)$, that the system be situated in the state with energy $E_\kappa(N, \Lambda)$ is computed by the formula

$$P_\kappa(N, \Lambda) = Q(N, \Lambda, \beta)^{-1} \exp[-\beta E_\kappa(N, \Lambda)] \tag{2.4}$$

Consider the following distribution function:

$$F_{N,\Lambda}(x) = \sum_{[E_\kappa(N,\Lambda) - \langle H_{N,\Lambda}\rangle]/|\Lambda|^{1/2} < x} P_\kappa(N, \Lambda) \tag{2.5}$$

where $|\Lambda|$ is the volume Λ. We shall demonstrate that under some hypotheses the distribution $F_{N,\Lambda}(x)$ tends to some gaussian distribution if $\Lambda \to \infty$ in the sense of Ref. 10 (p. 30), $|\Lambda|^{-1} N \to d$. We shall assume that the potential $V(x)$ belongs to one of the following classes:

(A) $V(x)$ is a continuous function bounded from below for $|x| > c_0 > 0$, $V(x) = +\infty$ if $|x| \leqslant c_0$, i.e., V has a hard core of radius c_0, and there exists an $\alpha_0 > 0$ and monotonically decreasing function $V_1(\imath)$, $\imath \geqslant \alpha_0$, such that $|V(x)| \leqslant V_1(|x|)$ if $|x| \geqslant \imath_0$,

$$\int_{\alpha_0}^{\infty} V_1(\imath)\, \imath^{\nu-1}\, d\imath < +\infty \tag{2.6}$$

(B) $V(x)$ is a continuous, nonnegative function for $|x| > 0$, and

$$\int_{|x| > \alpha_0} |V(x)|\, dx < +\infty, \qquad \text{where} \quad \alpha_0 > 0$$

Note that under our hypotheses there is a constant $B > 0$ such that if $|x_i - x_j| \geqslant c_0$, $i, j = 1, 2, ..., m$, $i \neq j$, then

$$\sum_j V(x_0 - x_j) \geqslant -B \tag{2.7}$$

Let us consider the following quantity:

$$\langle (H_{N,\Lambda} - \langle H_{N,\Lambda}\rangle)^2 \rangle = Q(N, \Lambda, \beta)^{-1} \sum_\kappa [E_\kappa(N, \Lambda) - \langle H_{N,\Lambda}\rangle]^2 \exp[-\beta E_\kappa(N, \Lambda)] \tag{2.8}$$

Theorem 1. Let $\nu = 1$, $V \in (A)$, and let $V(x)$ be finite, $\Lambda \to \infty$, $|\Lambda|^{-1} N \to d < c_0^{-1}$. Then

$$\lim |\Lambda|^{-1} \langle H_{N,\Lambda}\rangle = h_0, \qquad \lim |\Lambda|^{-1} \langle (H_{N,\Lambda} - \langle H_{N,\Lambda}\rangle)^2\rangle = g_0 \tag{2.9}$$

and if $g_0 \neq 0$ then

$$\lim F_{N,\Lambda}(x) = (2\pi g_0)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2g_0)\, dt \tag{2.10}$$

Theorem 2. Let v be arbitrary, $V \in (A)$, or $V \in (B)$, $\Lambda \to \infty$, $|\Lambda|^{-1} N \to d$. Then there is an $\varepsilon_0 > 0$ such that if $d < \varepsilon_0$ then

$$\lim |\Lambda|^{-1} \langle H_{N,\Lambda} \rangle = h_0, \qquad \lim |\Lambda|^{-1} \langle (H_{N,\Lambda} - \langle H_{N,\Lambda} \rangle)^2 \rangle = g_0 > 0 \quad (2.11)$$

$$\lim F_{N,\Lambda}(x) = (2\pi g_0)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2g_0)\, dt \quad (2.12)$$

Let us consider the grand canonical ensemble. Let $\mathscr{H}_{\pm}(\Lambda) = \oplus L^2_{\pm}(\Lambda^N)$ be the Fock space over Λ, N be the operator of the number of particles, $H_\Lambda = \oplus H_{N,\Lambda}$ be the Hamiltonian. Let

$$\langle H_\Lambda \rangle = Z(\Lambda, z, \beta)^{-1} \sum_N z^N \sum_\kappa E_\kappa(N, \Lambda) \exp[-\beta E_\kappa(N, \Lambda)] \quad (2.13)$$

$$\langle (H_\Lambda - \langle H_\Lambda \rangle)^2 \rangle = Z(\Lambda, z, \beta)^{-1} \sum_N z^N \sum_\kappa [E_\kappa(N, \Lambda) - \langle H_{N,\Lambda} \rangle]^2 \times \exp[-\beta E_\kappa(N, \Lambda)] \quad (2.14)$$

$$F_\Lambda(x) = Z(\Lambda, z, \beta)^{-1} \sum_N z^N \sum_{[E_\kappa(N,\Lambda) - \langle H_\Lambda \rangle]/|\Lambda|^{1/2} < x} \exp[-\beta E_\kappa(N, \Lambda)] \quad (2.15)$$

where $z > 0$ is the activity, and

$$Z(\Lambda, z, \beta) = \sum_N z^N \sum_\kappa \exp[-\beta E_\kappa(N, \Lambda)] \quad (2.16)$$

is the grand partition function.

Theorem 4. Let V be the same as in the Theorem 1. Then

$$\lim_{\Lambda \to \infty} |\Lambda|^{-1} \langle H_\Lambda \rangle = \tilde{h}_0, \qquad \lim_{\Lambda \to \infty} |\Lambda|^{-1} \langle (H_\Lambda - \langle H_\Lambda \rangle)^2 \rangle = \tilde{g}_0 \quad (2.17)$$

and if $\tilde{g}_0 \neq 0$, then

$$\lim F_\Lambda(x) = (2\pi \tilde{g}_0)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2\tilde{g}_0)\, dt \quad (2.18)$$

Theorem 4. Let V be the same as in Theorem 2. One can find $z_0 > 0$ such that for $0 < z < z_0$

$$\lim |\Lambda|^{-1} \langle H_\Lambda \rangle = \tilde{h}_0, \qquad \lim |\Lambda|^{-1} \langle (H_\Lambda - \langle H_\Lambda \rangle)^2 \rangle = \tilde{g}_0 > 0 \quad (2.19)$$

$$\lim F_\Lambda(x) = (2\pi \tilde{g}_0)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2g_0)\, dt \quad (2.20)$$

The proof of Theorems 1–4 is based on results related to the reduced density matrices of the systems, which were obtained in Refs. 1–6. Let

$\rho_\Lambda(x^m, y^m)$ denote the m-point reduced density matrix of the system, which is defined as (see Ref. I, p. 243)

$$\rho_\Lambda(x^m, y^m) = Z(\Lambda, z, \beta)^{-1} \sum_{\delta=0}^{\infty} \frac{z^{m+s}}{s!} \int_{\Lambda^s} du^s$$

$$\times \sum_{\Pi} (\pm)^{\Pi} \cdot \exp[-\beta H_{m+s,\Lambda}] \qquad [x^m u^s, \Pi(y^m u^s)] \qquad (2.21)$$

where $x^m = (x_1,..., x_m) \in R^{\nu m}$, $y^m = (y_1,..., y_m) \in R^{\nu m}$, $u^s = (u_1,..., u_s) \in \Lambda^s$, $\exp(-\beta H_{N,\Lambda}^{(0)})(x^N, y^N)$ denotes the kernel of the operator $\exp(-\beta H_{N,\Lambda}^{(0)})$, $H_{N,\Lambda}^{(0)}$ is the self-adjoint operator, defined by equation (2.1), and null boundary conditions on the space $L^2(\Lambda^N)$, the second sum runs over the elements of the symmetric group S_{m+s}, $(\pm)^{\Pi}$ is $+1$ for Bose statistics, and the signature of the permutation Π for Fermi statistics. We shall obtain the following results concerning the reduced density matrices.

Theorem 5. Let V be as in Theorem 1. Then the derivatives $\partial^{r+\kappa}\rho_\Lambda(x, y)/\partial\beta^r \, \partial z^k$ exist and

$$\lim_{\Lambda\to\infty} \frac{\partial^{r+\kappa}\rho_\Lambda(x, y)}{\partial\beta^r \, \partial z^\kappa} = \frac{\partial^{r+\kappa}\rho(x, y)}{\partial\beta^r \, \partial z^\kappa}$$

$$\lim_{\Lambda\to\infty} |\Lambda|^{-1} \int_\Lambda \frac{\partial^{r+\kappa}}{\partial\beta^r \, \partial z^\kappa} z^{-1}\rho_\Lambda(x, x) \, dx = \frac{\partial^{r+\kappa} z^{-1}\rho(0, 0)}{\partial\beta^r \, \partial z^\kappa} \qquad (2.22)$$

where $\rho(x^m, y^m)$ are the limiting reduced density matrices $m, \kappa, r = 1, 2,...$.

Theorem 6. Let V be the same as in Theorem 2. Then there is a $z_0 > 0$ such that for $|z| < z_0$ the derivatives $\partial^{r+\kappa}\rho_\Lambda(x, y)/\partial\beta^r \, \partial z^\kappa$ exist and

$$\lim_{\Lambda\to\infty} \frac{\partial^{r+\kappa}\rho_\Lambda(x^m, y^m)}{\partial\beta^r \, \partial z^\kappa} = \frac{\partial^{r+\kappa}\rho(x^m, y^m)}{\partial\beta^r \, \partial z^\kappa}$$

$$\lim_{\Lambda\to\infty} |\Lambda|^{-1} \int_\Lambda \frac{\partial^{r+\kappa}}{\partial\beta^r \, \partial z^\kappa} z^{-1}\rho_\Lambda(x, x) \, dx = \frac{\partial^{r+\kappa} z^{-1}\rho(0, 0)}{\partial\beta^r \, \partial z^\kappa} \qquad (2.23)$$

where $r = 1, 2, 3$, $m, \kappa = 1, 2,...$.

The proof of Theorems 5 and 6 presents specific technical difficulties. Namely, it is well known that the kernel $\exp(-\beta H_N^{(0)})(x^N, y^N)$ can be written by means of the Feinman–Kac formula (see Ref. I, p. 249)

$$\exp(-\beta H_{N,\Lambda}^{(0)})(x^N, y^N) = \int P_{x^N, y^N}^{\beta}(d\omega^N) \prod_{j=1}^{N} \alpha_\Lambda(\omega_j^N)$$

$$\times \exp\left[-\int_0^\beta U(\omega^N(s)) \, ds\right] \qquad (2.24)$$

where $P^{\beta}_{u,v}(d\eta)$ is the conditional Wiener measure, on the trajectories $\eta(S)$, $\alpha_{\Lambda}(\eta)$ the indicator functional of the set of all trajectories which lie in Λ, $x^N=(x_1,..., x_N)$, $y^N=(y_1,..., y_N)$, $\omega^N=(\omega_1,..., \omega_N)$

$$U(w^N)=\sum_{i\neq j} V(w_i-w_j) \tag{2.25}$$

Thus we must compute the derivatives of the integrals of type (2.24) with respect to parameter β. Integrals of such type were studied in the works,[11,12] but their results are insufficient for our purposes. In Section 3 we shall obtain the necessary results, concerning these derivatives.

3. DERIVATIVES OF THE WIENER INTEGRALS WITH RESPECT TO PARAMETER

Let $x, y\in R^q$, $\beta>0$, $\Omega(x, y, \beta)$ be the set of all continuous trajectories $\omega(s)$ in R^q, $0\leqslant s\leqslant\beta$ such that $\omega(0)=x$, $\omega(\beta)=y$; $P^{\beta}_{x,y}(d\omega)$ is a conditional Wiener measure on $\Omega(x, y, \beta)$ with the total mass $p(x, y, \beta)$, where $p(x, y, \beta)=(2\pi\beta)^{-q/2}\exp[-(x-y)^2/2\beta]$ is the Wiener transition function. Let Λ be a region in R^q, $U(z)$ a continuous, bounded from below real function on Λ. Put

$$K(x, y, \beta)=\int P^{\beta}_{x,y}(d\omega)\,\alpha_{\Lambda}(\omega)\exp\left[-\int_0^{\beta} U(\omega(s))\,ds\right] \tag{3.1}$$

where

$$\begin{aligned} &\alpha_{\Lambda}(\omega)=1 \text{ if } \omega(s)\in\Lambda \qquad \text{for all } s\in[0, \beta] \\ &\alpha_{\Lambda}(\omega)=0 \text{ otherwise} \end{aligned} \tag{3.2}$$

Lemma 3.1. The function $K(x, y, \beta)$ has an analytical extension with respect to the parameter β, in the domain $\operatorname{Re}\beta>0$, and moreover

$$|K(x, y, \beta)|\leqslant\exp(\operatorname{Re}\beta\cdot B)(2\pi\operatorname{Re}\beta)^{-q/2} \tag{3.3}$$

where $B=\inf U(z)$.

Proof. Let the domain Λ be bounded. Let H be a self-adjoint operator, defined by the expression $H=-\Delta+U(z)$, where Δ is Laplacian, with null boundary conditions. Then, due to the Feinman–Kac formula $K(x, y, \beta)=\exp(-\beta H)(x, y)$. By using the Mercer theorem, we can write the continuous positive definite kernel $K(x, y, \beta)$ in the form (see Ref. 13, p. 326)

$$K(x, y, \beta)=\sum\exp(-\beta E_{\kappa})\,\varphi_{\kappa}(x)\,\overline{\varphi_{\kappa}(y)} \tag{3.4}$$

where E_κ are eigenvalues of the operator H and φ_κ are the corresponding eigenfunctions $k = 1, 2, \ldots$. If $\operatorname{Re}\beta > 0$ then

$$\begin{aligned}
&\sum |\exp(-\beta E_\kappa)\,\varphi_\kappa(x)\,\overline{\varphi_\kappa(y)}| \\
&\quad \leqslant \left(\sum_\kappa \exp(-\operatorname{Re}\beta E_\kappa)\,|\varphi_\kappa(x)|^2\right)^{1/2}\left(\sum \exp(-\operatorname{Re}\beta E_\kappa)\,|\varphi_\kappa(y)|^2\right)^{1/2} \\
&\quad = [K(x, x, \operatorname{Re}\beta)]^{1/2}[K(y, y, \operatorname{Re}\beta)]^{1/2} \\
&\quad \leqslant \exp(\operatorname{Re}\beta B)\cdot(2\pi\operatorname{Re}\beta)^{-q/2}
\end{aligned} \tag{3.5}$$

If the region Λ is unbounded, we can use the compactness principle for the families of analytical functions. The Lemma is proved. ∎

Corollary 3.2. The function $K(x, y, \beta)$ has derivatives $\partial^r K(x, y, \beta)/\partial\beta^r$, $r = 1, 2, \ldots$, and the following estimates hold for $\beta > 0$:

$$\left|\frac{\partial^r K(x, y, \beta)}{\partial\beta^r}\right| \leqslant c_r^q \beta^{-r}\exp(2\beta B)\; p(x, y, \Lambda_{\beta y}) \tag{3.6}$$

where $B = \inf U(z)$, c_r constant.

Proof. By using the estimate (3.3), and Cauchy inequalities for the derivatives of an analytical function we can write

$$\left|\frac{\partial^r K(x, y, \beta)}{\partial\beta^r}\right| \leqslant r!\left(\frac{\beta}{2}\right)^{-r}\cdot\exp(2\beta B)(\pi\beta)^{-q/2} \tag{3.7}$$

Further,

$$\begin{aligned}
\left|\frac{\partial K(x, y, \beta)|}{\partial\beta}\right| &\leqslant \delta^{-1}(|K(x, y, \beta+\delta)| + |K(x, y, \beta)|) \\
&\quad + \max_{|\gamma-\beta|<\delta} |\partial^2 K(x, y, \gamma)/\partial\gamma^2|
\end{aligned} \tag{3.8}$$

where $\delta > 0$. Put $\delta = \exp[-(x-y)^2/4\beta]$. By using (3.7) and (3.8) we can obtain the estimate in the needed form, for the first derivative $\partial K(x, y, \beta)/\partial\beta$. Analogously, one can obtain the estimates (3.6) for the next derivatives. ∎

Let x, y, Λ be the same as above, $\mu_1 \geqslant 0$, $\mu_2 \geqslant 0, \ldots, \mu_q \geqslant 0$ are integers, let $U(x_{10}, x_{11}, \ldots, x_{1\mu_1}, x_{20}, x_{21}, \ldots, x_{2\mu_2}, \ldots, x_{q\mu_q})$ be a continuous bounded from below real function on $\Lambda \subset R^{\Sigma \mu_i + 1}$, and $W_1(z)$, $W_2(z)$,..., $W_n(z)$ bounded continuous, integrable functions on Λ. Consider the following integral:

$$L(x, y, \beta) = \iint \cdots \int P_{x_1, y_1}^{(1+\mu_1)\beta}(d\omega_1)\, P_{x_2, y_2}^{(1+\mu_2)\beta}(d\omega_2) \cdots P_{x_q, y_q}^{(1+\mu_q)\beta}(d\omega_q)$$
$$\times\, \alpha_\Lambda(\omega_1(\cdot), \omega_1(\cdot + \beta),..., \omega_1(\cdot + \mu_1 \beta), \omega_2(\cdot),$$
$$\omega_2(\cdot + \beta),..., \omega_2(\cdot + \mu_2 \beta),..., \omega_q(\cdot + \mu_q \beta))$$
$$\times \prod_{i=1}^{w} \left[\int_0^\beta W_i(\omega_1(s + k_{i1}\beta), \omega_2(s + k_{i2}\beta),..., \omega_q(s + k_{iq}\beta))\, ds \right]$$
$$\times \exp\left[- \int_0^\beta U(\omega_1(s), \omega_1(s+\beta),..., \omega_q(s + \mu_q \beta))\, ds \right] \tag{3.9}$$

where $0 \leqslant k_{ij} \leqslant \mu_j$ are some integers, $\beta > 0$.

Lemma 3.3. The function $L(x, y, \beta)$ has derivatives $\partial^r L/\partial \beta^r$ $r = 1$, 2,..., and the following estimates hold:

$$\left| \frac{\partial^r L(x, y, \beta)}{\partial \beta^r} \right|$$
$$\leqslant (\tilde{c}_r)^{q + \Sigma_{i=1}^q \mu_i} \eta^r \cdot \beta^{-r} \exp(2\beta B) \int_{R^1} \int_{R^1} \cdots \int_{R^1} dz_{11}\, dz_{12} \cdots dz_{1\mu_1}$$
$$\times\, dz_{21} \cdots \cdots dz_{q\mu_q} \sum_{g_1,...,g_n} \int_0^\beta ds_i \int_{s_1}^\beta ds_2 \cdots \int_{s_{n-1}}^\beta ds_n$$
$$\times \sum_{m_1 < m_2 < \cdots < m_r} \overbrace{\int_{R^{\mu_1+1}} \cdots \int_{R^{\mu_1+1}}}^{n} \overbrace{\int_{R^{\mu_2+1}} \cdots \int_{R^{\mu_2+1}}}^{n} \cdots$$
$$\times \overbrace{\int_{R^{\mu_q+1}} \cdots \int_{R^{\mu_q+1}}}^{n} du^{(1)} \cdots du^{(n)} \cdots dv^{(1)} \cdots dv^{(n)} \cdots \cdots dw^{(1)} \cdots dw^{(n)}$$
$$\times \prod_{j=1}^{w} |W_{g_j}(u_{\kappa_{g_j,1}}^{(j)}, v_{\kappa_{g_j,2}}^{(j)},..., w_{\kappa_{g_j,q}}^{(j)})|$$
$$\times \left[p(x_1, u_1^{(1)}, s_1) \prod_{j=2}^{\mu_1+1} p(z_{1j}, u_j^{(1)}, s_1) \right]$$
$$\times \left[\prod_{j=1}^{\mu_1+1} p(u_j^{(1)}, u_j^{(2)}, (s_2 - s_1)) \right] \cdots$$

$$\times \left[\prod_{j=1}^{\mu_1+1} p(u_j^{(m_1-2)}, u_j^{(m_1-1)}, (s_{m_1-1}-s_{m_1-2})) \right]$$
$$\times \left[\prod_{j=1}^{\mu_1+1} p(u_j^{(m_1-1)}, u_j^{(m_1)}, 4r(s_{m_1}-s_{m_1-1})) \right] \cdots$$
$$\times \left[p(x_2, v_1^{(1)}, s_1) \prod_{j=2}^{\mu_2+1} p(z_{2j}, v_j^{(1)}, s_1) \right]$$
$$\times \left[\prod_{j=1}^{\mu_2+1} p(v_j^{(1)}, v_j^{(2)}, (s_2-s_1)) \right] \cdots \tag{3.10}$$

where $\tilde{c}_r$ are constants, $g_i \neq g_j$, and if for example $\mu_1 = 0$ then the corresponding integration is absent.

Proof. We consider for simplicity the case $\mu_1 = \mu_2 = \cdots = 0$. Let

$$\hat{L}(x, y, \beta) = \int_0^\beta ds_1 \int_{s_1}^\beta ds_2 \cdots \int_{s_{n-1}}^\beta ds_n \int P_{x,y}^\beta(d\omega)\, \alpha_A(\omega) \times \prod_{j=1}^{n} W_j(\omega(s_j)) \exp\left[-\int_0^\beta U(\omega(s))\, ds \right] \tag{3.11}$$

Let us estimate the derivatives $\partial^r \hat{L}/\partial\beta^r$. Let $\varepsilon = \bar{\beta}/n+1$. We have

$$\hat{L}(x, y, \beta) = \Bigg[\int_\varepsilon^\beta ds_1 \int_{s_i}^\beta ds_2 \cdots \int_{s_{n-1}}^\beta ds_n + \int_0^\varepsilon ds_1 \int_{s_1+\varepsilon}^\beta ds_2 \int_{s_2}^\beta ds_3 \cdots \times \int_{s_{n-1}}^\beta ds_n + \cdots + \int_0^\varepsilon ds_1 \int_{s_1}^{s_1+\varepsilon} ds_2 \cdots \int_{s_{n-2}}^{s_{n-2}+\varepsilon} ds_{n-1} \int_{s_{n-1}+\varepsilon}^\beta ds_n$$
$$+ \int_0^\varepsilon ds_1 \int_{s_1}^{s_1+\varepsilon} ds_2 \cdots \int_{s_{n-2}}^{s_{n-2}+\varepsilon} ds_n \Bigg] \int P_{x,y}^\beta(d\omega)\, \alpha_A(\omega) \times W_1(\omega(s_1))\, W_2(\omega(s_2)) \cdots W_n(\omega(s_n)) \times \exp\left[-\int_0^\beta U(\omega(s))\, ds \right] \tag{3.12}$$

Let $x_1, \dots, x_m \in R^q$, $0 < s_1 < \cdots < s_m < \beta$, $\omega_0 \in \Omega(x, x_1, s_1)$, $\omega_1 \in \Omega(x_1, x_2, s_2 - s_1), \dots, \omega_m \in \Omega(x_m, y, \beta - s_m)$ and $(\omega_0, \dots, \omega_m)$ denote the trajectory $\omega \in \Omega(x, y, \beta)$ such, that

$$\omega(s) = \omega_0(s) \text{ for } 0 \leqslant s \leqslant s_1, \qquad \omega(s) = \omega_j(s_{j+1} - s) \text{ for } s_j \leqslant s \leqslant s_{j+1}$$
$$j = 1, 2, \dots, m-1, \qquad \omega(s) = \omega_m(\beta - s_m) \text{ for } s_m \leqslant s \leqslant \beta \tag{3.13}$$

Then for any functional $f(\omega)$, which is integrable with respect to $P^{\beta}_{x,y}(d\omega)$, the following formula:

$$\int P^{\beta}_{x,y}(d\omega)\, f(\omega)$$

$$= \int_{R^q} \int_{R^q} \cdots \int_{R^q} dx_1\, dx_2 \cdots dx_m$$

$$\times \int_{\Omega(x, x_1, s_1)} \int_{\Omega(x_1, x_2, s_2 - s_1)} \cdots \int_{\Omega(x_{m-1}, x_m, s_m - s_{m-1})}$$

$$\times \int_{\Omega(x_m, y, \beta - s_m)} P^{s_1}_{x_1, x_1}(d\omega_0)\, P^{s_2 - s_1}_{x_1, x_2}(d\omega_1) \cdots P^{s_m - s_{m-1}}_{x_{m-1}, x_m}(d\omega_{m-1})$$

$$\times P^{\beta - s_m}_{x_m, y}(d\omega_m)\, f((\omega_0, \omega_1, ..., \omega_m)) \tag{3.14}$$

holds. By using this formula, we obtain

$$\int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1 + \varepsilon} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2} + \varepsilon} ds_{m-1} \int_{s_{m-1} + \varepsilon}^{\beta} ds_m \int_{s_m}^{\beta} ds_{m+1} \cdots$$

$$\times \int_{s_{n-1}}^{\beta} ds_n \int P^{\beta}_{x,y}(d\omega)\, \alpha_A(\omega)$$

$$\times W_1(\omega(s_1))\, W_2(\omega(s_2)) \cdots W_n(\omega(s_n)) \exp\left[- \int_0^{\beta} U(\omega(s))\, ds \right]$$

$$= \int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1 + \varepsilon} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2} + \varepsilon} ds_{m-1} \int_{s_{m-1} + \varepsilon}^{\beta} ds_m \int_{s_m}^{\beta} ds_{m+1} \cdots \int_{s_{n-1}}^{\beta} ds_n$$

$$\times \int_{R^q} \int_{R^q} \cdots \int_{R^q} dx_1\, dx_2 \cdots dx_m\, W_1(x_1)\, W_2(x_2) \cdots W_m(x_m)$$

$$\times \iint \cdots \int P^{s_1}_{x,x}(d\omega_1)\, P^{s_2 - s_1}_{x_1, x_2}(d\omega_2) \cdots$$

$$\times P^{s_m - s_{m-1}}_{x_{m-1}, x_m}(d\omega_n)\, P^{\beta - s_n}_{x_n, y}(d\eta)\, \alpha_A(\omega_1)$$

$$\times \alpha_A(\omega_2) \cdots \alpha_A(\omega_n)\, \alpha_A(\eta)\, W_{m+1}(\eta(s_{m+1} - s_m))$$

$$\times W_{m+2}(\eta(s_{m+2} - s_m)) \cdots W_n(\omega(s_n - s_m))$$

$$= \exp\left[- \int_0^{s_1} U(\omega_1(t))\, dt - \int_0^{s_2 - s_1} U(\omega_2(t))\, dt - \cdots \right.$$

$$\left. - \int_0^{s_m - s_{m-1}} U(\omega_m(t))\, dt - \int_0^{\beta - s_m} U(\eta(t))\, dt \right]$$

$$= \int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1+\varepsilon_1} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\varepsilon} ds_{m-1} \int_{s_{m-1}+\varepsilon}^{\beta} ds_m \int dx_1\, dx_2 \cdots dx_m$$
$$\times W_1(x_1)\, W_2(x_2) \cdots W_m(x_m)$$
$$\times K(x, x_1, s_1)\, K(x_1, x_2, s_2 - s_1) \cdots$$
$$\times K(x_{m-1}, x_m, s_m - s_{m-1})\, \hat{L}_m(x_m, y, \beta - s_m) \tag{3.15}$$

where the function K is defined by (3.1), and

$$\hat{L}_m(\xi, \eta, \tau) = \int P^{\tau}_{\xi,\eta}(d\omega)\, \alpha_A(\omega) \int_0^{\tau} ds_1 \int_{s_1}^{\tau} ds_2 \cdots$$
$$\times \int_{s_{n-m-1}}^{\tau} ds_{n-m}\, W_{m+1}(\omega_1(s_1)) \cdots W_n(\omega(s_{n-m}))$$
$$\times \exp\left[- \int_0^{\tau} U(\omega(s))\, ds \right]$$

$\xi, \eta \in R^q$, $\tau > 0$. Put $s_m = \beta - t$ in (3.15). Then

$$\frac{\partial}{\partial \beta} \int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1+\varepsilon} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\varepsilon} ds_{m-1} \int_{s_{m-1}+\varepsilon}^{\beta} ds_m \int_{s_m}^{\beta} ds_{m+1} \cdots$$
$$\times \int_{s_{n-1}}^{\beta} ds_n \int P^{\beta}_{x,y}(d\omega)\, \alpha_A(\omega)$$
$$\times W_1(\omega(s_1)\, W_2(\omega(s_2)) \cdots W_n(\omega(s_n)) \exp\left[- \int_0^{\beta} U(\omega(s))\, ds \right]$$
$$= \int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1+\varepsilon} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\varepsilon} ds_{m-1} \int_0^{\beta - s_{m-1} - \varepsilon} dt$$
$$\times \iint \cdots \int dx_1\, dx_2 \cdots dx_m\, W_1(x_1) \cdots W_m(x_m)$$
$$\times K(x, x_1, s_1)\, K(x_1, x_2, s_2 - s_1) \cdots$$
$$\times \frac{\partial}{\partial \beta} K(x_{m-1}, x_m, \beta - t - s_{m-1})\, \hat{L}_m + \int_0^{\varepsilon} ds_1 \int_{s_1}^{s_1+\varepsilon} ds_2 \cdots \int_{s_{m-2}}^{s_{m-2}+\varepsilon}$$
$$\times \iint \cdots \int dx_1\, dx_2 \cdots dx_m\, W_1(x_1)\, W_2(x_2) \cdots W_m(x_m)$$
$$\times K(x, x_1, s_1)\, K(x_1, x_2, s_2 - s_1) \cdots K(x_{m-2}, x_{m-1}, s_{m-1} - s_{m-2})$$
$$\times K(x_{m-1}, x_m, \varepsilon) \cdot \hat{L}_m(x_m, y, \beta - s_{m-1} - \varepsilon) \tag{3.16}$$

Note, that $\beta - t - s_{m-1} > \varepsilon$ in (3.16). Hence, by Corollary 3.2

$$\left|\frac{\partial K(x_{m-1}, x_m, \beta - t - s_{m-1})}{\partial \beta}\right| \leqslant c_1^q \varepsilon^{-1} \exp[2(\beta - t - s_{m-1})B]\, p(x_{m-1}, x_m, 4(\beta - t - s_{m-1})) \quad (3.17)$$

In addition

$$K(u, v, \tau) \leqslant \exp(\tau B)\, p(u, v, \tau)$$

$$\begin{aligned}|\hat{L}_m(u, v, \tau)| \leqslant \exp(\tau B) \int_0^\tau ds_{m+1} \int_{s_{m+1}}^\tau ds_{m+2} \cdots \\ \times \int_{s_{n-1}}^\tau ds_n \int_A \cdots \int_A dx_{m+1} \cdots dx_n \\ \times\ p(u, x_{m+1}, s_{m+1})\, p(x_{m+1}, x_{m+2}, s_{m+2} - s_{m+1}) \cdots \\ \times\ p(x_n, v, \tau - s_n)\end{aligned} \quad (3.18)$$

From (3.12)–(3.18) we have

$$\begin{aligned}\left|\frac{\partial \hat{L}(x, y, \beta)}{\partial \beta}\right| \leqslant \tilde{c}_1^q \varepsilon^{-1} \exp(2\beta B) \int_0^\beta ds_1 \int_{s_1}^\beta ds_2 \cdots \int_{s_{n-1}}^\beta ds_n \\ \times \sum_m \iint \cdots \int dx_1 \cdots dx_n \\ \times \prod_{i=1}^n |W_i(x_i)|\, p(x, x_1, s_1)\, p(x_1, x_2, s_2 - s_1) \cdots \\ \times\ p(x_{m-2}, x_{m-1}, s_{m-1} - s_{m-2})\, p(x_{m-1}, x_m, 4(s_m - s_{m-1})) \\ \times\ p(x_m, x_{m+1}, s_{m+1} - s_m) \cdots p(x_n, y, \beta - s_n)\end{aligned} \quad (3.19)$$

where $\tilde{c}_1$ is a constant. The necessary estimate for the derivative $\partial L/\partial \beta$ follows immediately from (3.19). Further, we can differentiate the relation (3.16) with respect to β, and by means of (3.19) get the estimate

$$\begin{aligned}\left|\frac{\partial^2 \hat{L}(x, y, \beta)}{\partial \beta^2}\right| \\ \leqslant c_2^q \varepsilon^{-2} \exp(2\beta B) \int_0^\beta ds_1 \int_{s_1}^\beta ds_2 \cdots \int_{s_{n-1}}^\beta ds_n \sum_{m_1 < m_2} \iint \cdots \int dx_1 \cdots dx_n \\ \times \prod_i W_i(x)\, p(x, x_1, s_1) \cdots p(x_{m_1-2}, x_{m_1-1}, s_{m_1-1} - s_{m_1-2}) \\ \times\ p(x_{m_1-1}, x_{m_1}, 8(s_{m_1} - s_{m_1-1}))\, p(x_{m_1}, x_{m_1+1}, s_{m_1+1} - s_{m_1}) \cdots \\ \times\ p(x_{m_2-1}, x_{m_2}, 8(s_{m_2} - s_{m_2-1}))\, p(x_{m_2}, x_{m_2+1}, s_{m_2+1} - s_{m_2}) \cdots\end{aligned}$$

from which immediately follows the necessary estimate for the derivative $\partial^2 L/\partial\beta^2$, and so on. Thus one can get the necessary estimates in the case $\mu_1 = \mu_2 = \cdots = \mu_q = 0$. Starting from this it is possible to deduce the analogous computations in the general case. The Lemma is proved. ∎

Lemma 3.4. The derivatives $\partial^r K(x, y, \beta)/\partial\beta^r$ can also be estimated in the following way:

$$\left|\frac{\partial^r K(x, y, \beta)}{\partial\beta^r}\right| \leqslant \lambda^r [K(x, x, \beta)]^{1/2} \left\{K(y, y, \beta) + (\pi\beta)^{-q/2} \exp\left[-\frac{\beta}{2}(\lambda - B)\right]\right\}^{1/2} \tag{3.20}$$

where $B = \inf U(z)$, $\lambda > \max(B, 4r\beta^{-1})$, $r = 1, 2, \ldots$.

Proof. Let H be the same as in the proof of Lemma 3.1. Let us write the function $K(x, y, \beta)$ in the form (3.4). For the eigenvalues E_κ of the operator H the inequality $E_\kappa \geqslant -B$ holds. Hence, we have for $\lambda > \max(B, 4r\beta^{-1})$

$$\begin{aligned}
&\left|\sum_\kappa E_\kappa^r \exp(-\beta E_\kappa)\, \varphi_\kappa(x)\, \overline{\varphi_\kappa(y)}\right| \\
&\quad \leqslant \left[\sum_\kappa \exp(-\beta E_\kappa)\, |\varphi_\kappa(x)|^2\right]^{1/2} \left[\sum_\kappa E_\kappa^{2r} \exp(-\beta E_\kappa)\, |\varphi_\kappa(y)|^2\right]^{1/2} \\
&\quad \leqslant [K(x, x, \beta)]^{1/2} \left[\lambda^{2r} \cdot \sum_{E_\kappa \leqslant \lambda} \exp(-\beta E_\kappa)\, |\varphi_\kappa(y)|^2\right. \\
&\qquad \left. + [\max_{E > \lambda} E^{2r} \exp(-\beta E/2)] \sum_{E_\kappa > \lambda} \exp[(-\beta E_\kappa/2)\, |\varphi_\kappa(y)|^2\right]^{1/2} \\
&\quad \leqslant \lambda^r [K(x, x, \beta)]^{1/2} [K(y, y, \beta) + \exp(-\beta\lambda/2)\, K(y, y, \beta/2)]^{1/2}
\end{aligned}$$

The Lemma is proved. ∎

4. PROOF OF THEOREMS 1, 3, AND 5

First we prove Theorem 5. To this end we now obtain some rough estimates for the derivatives of the reduced density matrices with respect to the parameter β. Let $V(x)$, $\rho_\Lambda(x^m, y^m)$ be the same as in Theorem 5.

Lemma 4.1. The functions $\rho_\Lambda(x^m, y^m)$ have derivatives $\partial^{r+\kappa}\rho_\Lambda(x^m, y^m)/\partial\beta^r\,\partial z^\kappa$ with respect to the parameters β, z, and the following estimates hold:

$$\left|\frac{\partial^{r+\kappa}}{\partial\beta^r\,\partial z^\kappa} z^{-m}\rho_\Lambda(x^m, y^m)\right| \leqslant c_{r,\kappa,m}(\beta, z)\,|\Lambda|^{2r} \tag{4.1}$$

where $c_{r,\kappa,m}(\beta, z)$ are constants, $\sup\{c_{r,\kappa}(\beta, z)\colon 0<\beta_1\leqslant\beta\leqslant\beta_2<+\infty,$ $0\leqslant z\leqslant z_1\}<+\infty$.

Proof. Let $n\leqslant|\Lambda|^{3/2}$. Let us estimate the derivative

$$\frac{\partial^r}{\partial\beta^r}\iint P^\beta_{x^m,y^m}(d\omega^m)\,P^\beta_{u^n,u^n}(d\eta^n)$$
$$\times\alpha_\Lambda(\omega^m,\eta^n)\exp\left[-\int_0^\beta U(\omega^m(s),\eta^n(s))\,ds\right] \tag{4.2}$$

by means of Lemma 3.4, where u^n, α_Λ, U are the same as in (2.24). As long as $U(w^m, y^n)\geqslant-(m+n)\,B_0$, $n\leqslant|\Lambda|^{3/2}$, we can apply Lemma 3.4 with $B=(m+n)\,B_0$, $\lambda=2mB_0\,|\Lambda|^2$. By using the Shwartz inequality we obtain

$$\sum_{n\leqslant|\Lambda|^{3/2}}\left|\frac{z^n}{n!}\int_{\Lambda^n}du^n\frac{\partial^r}{\partial\beta^r}\iint P^\beta_{x^m,y^m}(d\omega^m)\,P^\beta_{u^n,u^n}(d\eta^n)\right.$$
$$\left.\times\alpha_\Lambda(\omega^m,\eta^n)\exp\left[-\int_0^\beta U(\omega^m(s),\eta^n(s))\,ds\right]\right|$$
$$\leqslant 2mB_0\,|\Lambda|^{2r}\left[\sum_{n\leqslant|\Lambda|^{3/2}}\frac{|z|^n}{n!}\int_{\Lambda^n}du^n\iint P^\beta_{x^m,y^m}(d\omega^m)\,P^\beta_{u^n,u^n}(d\eta^n)\right.$$
$$\left.\times\alpha_\Lambda(\omega^m,\eta^n)\exp\left[-\int_0^\beta U(\omega^m(s),\eta^n(s))\,ds\right]\right]^{1/2}$$
$$\times\left\{\sum_{n\leqslant|\Lambda|^{3/2}}\frac{|z|^n}{n!}\left[\int_{\Lambda^n}du^n\iint P^\beta_{y^m,y^m}(d\omega^m)\,P^\beta_{u^n,u^n}(d\eta^n)\right.\right.$$
$$\times\alpha_\Lambda(\omega^m,\eta^n)\exp\left[-\int_0^\beta U(\omega^m(s),\eta^n(s))\,ds\right]$$
$$\left.\left.+(\pi\beta)^{-(n+m)/2}\,|\Lambda|^n\exp\left(-\frac{\beta}{2}|\Lambda|^2\right)\right]\right\}^{1/2}$$
$$\leqslant 2mB_0\exp(\beta mB_0)(2\pi\beta)^{-m/2}\cdot|z|^m\cdot|\Lambda|^{2r}\cdot Z(\Lambda,|z|,\beta) \tag{4.3}$$

where $Z(\Lambda,|z|,\beta)$ is the grand partition function.

If $n > |\Lambda|^{3/2}$, then we estimate the derivatives (4.2) by means of Lemma 3.1, and the Cauchy inequalities for the derivatives of an analytical function. We have

$$\sum_{n>|\Lambda|^{3/2}} \left| \frac{z^n}{n!} \int_{\Lambda^n} du^n \frac{\partial^r}{\partial\beta^r} \iint P^{\beta}_{x^m,y^m}(d\omega^m)\, P^{\beta}_{u^n,u^n}(d\eta^n)\, \alpha_\Lambda(\omega^m, \eta^n) \right.$$
$$\left. \times \exp\left[- \int_0^\beta U(\omega^m(s), \eta^n(s))\, ds \right] \right|$$
$$\leqslant r! \left(\frac{\beta}{2}\right)^{-r} \sum_{n>|\Lambda|^{3/2}} \frac{|z|^n}{n!} (\pi\beta)^{-(n+m)/2} \exp[2\beta(m+n)\, B_0]\, |\Lambda|^n$$
$$\leqslant r! \left(\frac{\beta}{2}\right)^{-r} (\pi\beta)^{-m/2} \exp(2\beta m B)$$
$$\times \sum_{n>|\Lambda|^{3/2}} \left(\frac{z\pi\beta\, |\Lambda| \exp(2\beta B_0 + 1)}{n} \right)^n \tag{4.4}$$

Let

$$Z(\Lambda, z, \beta, x^m, y^m) = \sum_{n=0}^{\infty} \frac{z^{n+m}}{n!} \int_{\Lambda^n} du^n \iint P^{\beta}_{x^m,y^m}(d\omega^m)\, P^{\beta}_{u^n,u^n}(d\eta^n)$$
$$\times\, \alpha_\Lambda(\omega^m, \eta^n) \exp\left[- \int_0^\beta U(\omega^m(s), \eta^n(s))\, ds \right] \tag{4.5}$$

It follows from (4.4), (4.3), that

$$\left| \frac{\partial^r}{\partial\beta^r} z^{-m} Z(\Lambda, z, \beta, x^m, y^m) \right| \leqslant \tilde{c}_{r,m}(\beta)\, |\Lambda|^{2r}\, Z(\Lambda, |z|, \beta) \tag{4.6}$$

where $\tilde{c}_{r,m}(\beta)$ is constant, $|\Lambda| > \lambda_{r,m}(\beta, z)$, and $\lambda_{r,m}(\beta, z)$ is constant. Analogously,

$$\left| \frac{\partial^r}{\partial\beta^r} Z(\Lambda, z, \beta) \right| \leqslant \tilde{c}_{r,m}(\beta)\, |\Lambda|^{2r}\, Z(\Lambda, |z|, \beta) \tag{4.7}$$

Hence, if $z > 0$ then

$$\left| \frac{\partial}{\partial\beta} z^{-m} \rho_\Lambda(x^m, y^m) \right| \leqslant \left| \frac{\partial}{\partial\beta} z^{-m} Z(\Lambda, z, \beta, x^m, y^m) \right| Z(\Lambda, z, \beta)^{-1}$$
$$+ z^{-m} Z(\Lambda, z, \beta, x^m, y^m) \left| \frac{\partial}{\partial\beta} Z(\Lambda, z, \beta) \right| Z(\Lambda, z, \beta)^{-2}$$
$$\leqslant c_{1,m}(\beta)\, |\Lambda|^{2r} \tag{4.8}$$

where $c_{1,m}(\beta)$ is a constant. Analogously, one can estimate the other derivatives.

Now we can prove Theorem 5. Let us consider for example the 1-point reduced density matrices $\rho_\Lambda(x, y)$. It follows from the results of Ref. 5 that one can find $s_1 = s_1(z, \beta) > 0$ such that if $x, y \in [a, b]$, $\Lambda = (a-s, b+t)$, $\Lambda' = (a-s', b+t)$, $\Lambda'' = (a-r, b+q)$, $\Lambda''' = (a-r, b+q')$, $s_1 \leqslant s \leqslant s' \leqslant s+1$, $s_1 \leqslant q \leqslant q' \leqslant q+1$ then for fixed $b-a$

$$|z|^{-1}\,|\rho_\Lambda(x, y) - \rho_{\Lambda'}(x, y)| < \exp(-s^{\alpha_1}) \tag{4.9}$$

$$|z|^{-1}\,|\rho_{\Lambda''}(x, y) - \rho_{\Lambda'''}(x, y)| < \exp(-t^{\alpha_1}) \tag{4.10}$$

where $\alpha_1 = \alpha_1(z, \beta) > 0$; moreover if $|z| \leqslant z_1$, $0 < \beta_1 < \beta < \beta_1' < +\infty$, then $\sup s_1(z, \beta) = \bar{s}_1 < +\infty$, $\inf \alpha_1(z, \beta) = \bar{\alpha}_1 > 0$. Hence we can write

$$\begin{aligned}
&|z|^{-1}\left|\frac{\partial}{\partial\beta}\rho_\Lambda(x, y) - \frac{\partial}{\partial\beta}\rho_{\Lambda'}(x, y)\right| \\
&\quad\leqslant |z|^{-1}\left|\frac{\partial}{\partial\beta}\rho_{\Lambda,\beta}(x, y) - \varepsilon^{-1}[\rho_{\Lambda,\beta+\varepsilon}(x, y) - \rho_{\Lambda,\beta}(x, y)]\right| \\
&\qquad+ \varepsilon^{-1}\,|z|^{-1} \max_{|\beta-\gamma|\leqslant\varepsilon} |\rho_{\Lambda,\gamma}(x, y) - \rho_{\Lambda',\gamma}(x, y)| \\
&\qquad+ |z|^{-1}\left|\frac{\partial}{\partial\beta}\rho_{\Lambda',\beta}(x, y) - \varepsilon^{-1}[\rho_{\Lambda,\beta+\varepsilon}(x, y) - \rho_{\Lambda,\beta}(x, y)]\right| \\
&\quad\leqslant \varepsilon\,|z|^{-1} \max_{|\beta-\gamma|<\varepsilon}\left|\frac{\partial^2}{\partial\gamma^2}\rho_{\Lambda,\beta}(x, y)\right| + \varepsilon^{-1}\,|z|^{-1}\exp(-s^{\bar{\alpha}_1}) \\
&\qquad+ \varepsilon\,|z|^{-1}\max_{|\beta-\gamma|<\varepsilon}\left|\frac{\partial^2}{\partial\gamma^2}\rho_{\Lambda',\beta}(x, y)\right|
\end{aligned} \tag{4.11}$$

Put $\varepsilon = \exp(-s^{\bar{\alpha}_1}/2)$. By using the estimates (4.9) and (4.1), we have

$$|z|^{-1}\left|\frac{\partial}{\partial\beta}\rho_\Lambda(x, y) - \frac{\partial}{\partial\beta}\rho_{\Lambda'}(x, y)\right| \leqslant \text{const}\,|\Lambda|^2 \exp\left(-\frac{s^{\bar{\alpha}_1}}{2}\right) \tag{4.12}$$

Analogously

$$|z|^{-1}\left|\frac{\partial}{\partial\beta}\rho_{\Lambda''}(x, y) - \frac{\partial}{\partial\beta}\rho_{\Lambda'''}(x, y)\right| \leqslant \text{const}\,|\Lambda|^2 \exp\left(-\frac{t^{\bar{\alpha}_1}}{2}\right) \tag{4.13}$$

Let Λ be the same as above and $s = t$, $\Omega = (a-s-k, b+s+q)$, $g \geqslant$

$k \geqslant 0$, q, κ integers. Put $\Omega_j = (a-s-j, b+s+j)$, $j=0, 1, 2,..., k$, $\Omega_{j+\kappa} = (a-s-\kappa, b+s+\kappa+j)$, $j=1, 2,..., q-\kappa$. Then, from (4.12), (4.13)

$$|z|^{-1} \left| \frac{\partial}{\partial\beta} \rho_\Lambda(x, y) - \frac{\partial}{\partial\beta} \rho_\Omega(x, y) \right|$$
$$\leqslant |z|^{-1} \sum_{j=0}^{q-1} \left| \frac{\partial}{\partial\beta} \rho_{\Omega_j}(x, y) - \frac{\partial}{\partial\beta} \rho_{\Omega_{j+1}}(x, y) \right|$$
$$\leqslant \text{const} \sum_{j=0}^{\kappa} (2s+2j+b-a)^2 \exp[-(s+j)^{\bar{\alpha}_1}/2]$$
$$\leqslant \text{const} \exp(-s^{\alpha_2}) \tag{4.14}$$

where $\alpha_2 > 0$ is a constant. If $s \neq t$ then

$$|z|^{-1} \left| \frac{\partial}{\partial\beta} \rho_\Lambda(x, y) - \frac{\partial}{\partial\beta} \rho_\Omega(x, y) \right| \leqslant \text{const} \exp\{-[\min(s, t)]^{\alpha_2}\} \tag{4.15}$$

Let $\rho(x, y)$ be the limiting 1-point reduced density matrix. It follows from the results of Ref. 4 and inequality (4.15) that the matrix $\rho(x, y)$ has a derivative $(\partial/\partial\beta)\,\rho(x, y)$ with respect to the parameter β, and

$$|z|^{-1} \left| \frac{\partial}{\partial\beta} \rho_\Lambda(x, y) - \frac{\partial}{\partial\beta} \rho(x, y) \right| \leqslant \text{const} \exp\{-[\min(s, t)]^{\alpha_2}\} \tag{4.16}$$

Let us verify that

$$|z|^{-1} \left| |\Lambda|^{-1} \int_\Lambda \frac{\partial}{\partial\beta} \rho_\Lambda(x, x)\, dx - \frac{\partial}{\partial\beta} \rho(0, 0) \right| \leqslant \varepsilon_\Lambda \tag{4.17}$$

where $\varepsilon_\Lambda \to \infty$ when $\Lambda \to \infty$. Let $\Lambda = (\gamma_1, \gamma_2)$, $\Lambda_1 = (\gamma_1 + r_\Lambda, \gamma_2 - r_\Lambda)$ where $r_\Lambda \to \infty$ when $\Lambda \to \infty$. Then from (4.15)

$$|z|^{-1}\, |\Lambda|^{-1} \left| \int_{\Lambda_1} \left[\frac{\partial}{\partial\beta} \rho_\Lambda(x, x) - \frac{\partial}{\partial\beta} \rho(0, 0) \right] dx \right| \leqslant \text{const} \exp(-r_\Lambda^{\alpha_2})$$

Let $\Lambda_2 = (\gamma_1, \gamma_1 + 2r_\Lambda)$. By using the estimate (4.13) we may verify, that for

$$|z|^{-1} \left| \frac{\partial}{\partial\beta} \rho_\Lambda(x, x) - \frac{\partial}{\partial\beta} \rho_{\Lambda_2}(x, x) \right| \leqslant \text{const} \exp(-r_\Lambda^{\alpha_2}) \tag{4.18}$$

It follows from (4.18) and Lemma 4.1 that

$$|z|^{-1} \left| \frac{\partial}{\partial\beta} \rho_\Lambda(x, x) \right| \leqslant \text{const}\, r_\Lambda^2 \tag{4.19}$$

Hence,

$$|z|^{-1}\,|\Lambda|^{-1}\int_{\gamma_1}^{\gamma_1+r_\Lambda}\left|\frac{\partial}{\partial\beta}\rho_\Lambda(x,x)-\frac{\partial}{\partial\beta}\rho(0,0)\right|dx\leqslant \text{const}\,|\Lambda|^{-1}\,r_\Lambda^3 \quad (4.20)$$

Analogously,

$$|z|^{-1}\,|\Lambda|^{-1}\int_{\gamma_2-r_\Lambda}^{\gamma_2}\left|\frac{\partial}{\partial\beta}\rho_\Lambda(x,x)-\frac{\partial}{\partial\beta}\rho(0,0)\right|dx\leqslant \text{const}\,|\Lambda|^{-1}\,r_\Lambda^3 \quad (4.21)$$

Thus we have obtained the relation (4.17).

Analogously, one can obtain the other statements of Theorem 5.

Let us prove Theorem 3. We have the following formulas:

$$\langle H_{N,\Lambda}\rangle = -\frac{\partial}{\partial\beta}\ln Q(\Lambda,N,\beta),\qquad \langle H_\Lambda\rangle = -\frac{\partial}{\partial\beta}\ln Z(\Lambda,z,\beta) \quad (4.22)$$

$$\begin{aligned}\langle (H_{N,\Lambda}-\langle H_{N,\Lambda}\rangle)^2\rangle &= -\frac{\partial^2}{\partial\beta^2}\ln Q(\Lambda,N,\beta)\\ \langle (H_\Lambda-\langle H_\Lambda\rangle)^2\rangle &= \frac{\partial^2}{\partial\beta^2}\ln Z(\Lambda,z,\beta)\end{aligned} \quad (4.23)$$

$$Z\frac{d}{dz}\ln Z(\Lambda,z,\beta)=\int_\Lambda \rho_\Lambda(x,x)\,dx \quad (4.24)$$

where $Q(\Lambda,N,\beta)$, $\langle H_{N,\Lambda}\rangle$, $\langle (H_{N,\Lambda}-\langle H_{N,\Lambda}\rangle)^2\rangle$, $Z(\Lambda,z,\beta)$, $\langle H_\Lambda\rangle$, $\langle (H_\Lambda-\langle H_\Lambda\rangle)^2\rangle$ are defined by (2.2), (2.3), (2.8), (2.13), (2.14), (2.16). It follows from Theorem 5 that under the hypotheses of Theorem 3

$$\lim_{\Lambda\to\infty}|\Lambda|^{-1}\frac{\partial^{r+\kappa}}{\partial\beta^r\,\partial z^\kappa}\ln Z(\Lambda,z,\beta)=\frac{\partial^{r+\kappa}\lambda(z,\beta)}{\partial\beta^r\,\partial z^\kappa} \quad (4.25)$$

uniformly for $|z|\leqslant c_1$, $\beta\in[\beta_1,\beta_2]$; where $\lambda(z,\beta)$ certain function c_1, β_1, $\beta_2>0$ arbitrary constants. ∎

Lemma 4.2. Let $Q(\Lambda,N,\beta)$, $Z(\Lambda,z,\beta)$ be the same as above. Let the functions $F_{N,\Lambda}(x)$, $F_\Lambda(x)$ be defined by (2.5) and (2.15), respectively. If for $r=1,2,3$

$$\lim|\Lambda|^{-1}\frac{\partial^r}{\partial\beta^r}\ln Q(\Lambda,N,\beta)=g_r \quad (4.26)$$

uniformly for $\beta\in[\beta_1,\beta_2]$, where $g_r(\beta)$ are bounded, $g_2(\beta)>0$, then

$$\lim F_{N,\Lambda}(x)=(2\pi g_2)^{-1/2}\int_{-\infty}^{x}\exp(-t^2/2g_2)\,dt \quad (4.27)$$

Analogously, if for $r=1, 2, 3$

$$\lim |\Lambda|^{-1} \frac{\partial^r}{\partial \beta^r} \ln Z(\Lambda, z, \beta) = g_r \tag{4.28}$$

uniformly for $\beta \in [\beta_1, \beta_2]$, where g_r are bounded, $g_2(\beta) > 0$ then

$$\lim F_\Lambda(x) = (2\pi g_2)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2g_2)\, dt \tag{4.29}$$

Proof. Note that for any function $f(x)$

$$\int_{-\infty}^{\infty} f(x)\, dG_{N,\Lambda}(x) = Q(N, \Lambda, \beta)^{-1} \sum_{\kappa} f\left(\frac{E_\kappa(N, \Lambda) - \langle H_{N,\Lambda}\rangle}{|\Lambda|^{1/2}}\right) \times \exp[-\beta E_\kappa(N, \Lambda)]$$

where $E_\kappa(N, \Lambda)$ are the same as in (2.2), in case the sum on the right is finite. Hence, for fixed s and $|\Lambda| > (s\beta^{-1})^2$

$$\theta_{N,\Lambda}(s) = \int_{-\infty}^{\infty} \exp(sx)\, dF_{N,\Lambda}(x) = \exp(-s\, |\Lambda|^{-1/2} \langle H_{N,\Lambda}\rangle) \times Q(\Lambda, N, \beta)^{-1}\, Q(\Lambda, N, \beta - s\, |\Lambda|^{-1/2}) < +\infty \tag{4.30}$$

Let us write the function $\theta_{N,\Lambda}(s)$ by means of the Taylor formula

$$\ln \theta_{N,\Lambda}(s) = -s\, |\Lambda|^{-1/2} \langle H_{N,\Lambda}\rangle + s\, |\Lambda|^{-1/2} \frac{\partial}{\partial \beta} \ln Q(\Lambda, N, \beta) + \frac{s^2\, |\Lambda|^{-1}}{2!} \frac{\partial^2}{\partial \beta^2} \ln Q(\Lambda, N, \beta) - \frac{s^3\, |\Lambda|^{3/2}}{3!} \frac{\partial^3}{\partial \theta^3} \ln Q(N, \Lambda, \theta) \tag{4.31}$$

where $|\theta - \beta| < |\Lambda|^{-1/2} s$. It follows from (4.30), (4.31), and (4.26) that

$$\lim \int_{-\infty}^{\infty} \exp(sx)\, dF_{N,\Lambda}(x) = \exp(g_2 s^2/2) \tag{4.32}$$

Thus Lemma 2 follows from the next Lemma 4.3. ∎

Lemma 3. Let $F_n(x)$ be distribution functions. Let for all $s \in [-a, b]$

$$\int_{-\infty}^{\infty} \exp(sx)\, dF_n(x) < +\infty, \qquad n \geqslant n_0 \tag{4.33}$$

and suppose that there exist the limits

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \exp(sx)\, dF_n(x) = \exp(gs^2/2) \tag{4.34}$$

where a, b, $g > 0$. Then

$$\lim_{n\to\infty} F_n(x) = (2\pi g)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2g) \tag{4.35}$$

Proof. Put

$$\theta_n(s+it) = \int_{-\infty}^{\infty} \exp[(s+it)x]\, dF_n(x), \qquad n \geqslant n_0 \tag{4.36}$$

where $s \in [-a, b]$, $t \in (-\infty, +\infty)$. It follows from (4.33), (4.34) that θ_n are analytical functions of the variable $z = s + it$, and $|\theta_n(z)| \leqslant M_0$, where $M_0 < +\infty$ is a constant. The family $\{\theta_n(z)\}$ is compact according to the compactness principle for families of analytical functions. It follows from the uniqueness theorem for analytical functions and relation (4.34) that

$$\lim_{n\to\infty} \theta_n(s+it) = \exp[g(s+it)^2/2]$$

In particular,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \exp(itx)\, dF_n(x) = \exp(-gt^2/2)$$

Hence, the relation (4.27) is correct. Analogously, one can obtain also the second statement of Lemma 4.2.

Thus we have obtained the proof of Theorem 3. Consider the function

$$\psi_{\Lambda,z,\beta}(\lambda) = Z(\Lambda, z, \beta)^{-1}\, Z(\Lambda, ze^{i\lambda}\beta) = \sum_{N=0}^{\infty} e^{iN\lambda} \frac{z^N}{N!} \frac{Q(N, \Lambda, \beta)}{Z(\Lambda, z, \beta)} \tag{4.37}$$

where z, $\beta > 0$, $\lambda \in (-\infty, \infty)$. We have

$$\frac{z^N}{N!} \frac{Q(N, \Lambda, \beta)}{Z(\Lambda, z, \beta)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iN\lambda} \psi_{\Lambda,z,\beta}(\lambda)\, d\lambda \tag{4.38}$$

In Ref. 8 it is shown that under the hypotheses of Theorem 1, the following estimate,

$$|\psi_{\Lambda,z,\beta}(\lambda)| \leqslant \exp(-|\Lambda|^{\alpha}), \qquad \text{if} \quad |\lambda| > |\Lambda|^{-\alpha} \tag{4.39}$$

holds for $\alpha > 0$. It follows from Lemma 4.1 that

$$\left| \frac{\partial^r}{\partial \beta^r} \psi_{\Lambda,z,\beta}(\lambda) \right| \leqslant c_{r,1}(\beta, z)\, |\Lambda|^{2r} \tag{4.40}$$

where $C_{r,1}(\beta)$ is a constant. We have

$$\left| \frac{\partial \psi_{\Lambda,z,\beta}}{\partial \beta} \right| \leqslant \varepsilon^{-1}\, |\psi_{\Lambda,z,\beta+\delta}| + \varepsilon^{-1}\, |\psi_{\Lambda,z,\beta}| + \varepsilon \max_{|\beta' - \beta| \leqslant \varepsilon} \left| \frac{\partial^2 \psi_{\Lambda,z,\beta}}{\partial^2 \beta} \bigg|_{\beta=\beta'} \right| \tag{4.41}$$

where $\varepsilon > 0$. If we take in (4.41) $|\lambda| > |\Lambda|^{-\alpha}$, $\varepsilon = \exp(-|\Lambda|^{\alpha}/2)$ then by using the estimate (4.39) we shall obtain

$$\left| \frac{\partial \psi_{\Lambda,z,\beta}(\lambda)}{\partial \beta} \right| < \tilde{c}_1(\beta, z) \exp(-|\Lambda|^{\alpha/3}) \tag{4.42}$$

where $\tilde{c}_1(\beta, z)$ is a constant.

Now we expound in powers of λ the expression $\ln \psi_{\Lambda,z,\beta}(\lambda)$ in the neighborhood $|\lambda| < |\Lambda|^{-\alpha}$. We have

$$\ln \psi_{\Lambda,z,\beta}(\lambda) = \lambda \frac{\partial}{\partial \lambda} \ln Z(\Lambda, ze^{i\lambda}, \beta)\, |_{\lambda=0} + \frac{\lambda^2}{2} \frac{\partial^2}{\partial \lambda^2} \ln Z(\Lambda, ze^{i\lambda}, \beta) + R \tag{4.43}$$

where R is the remainder. It follows from the definition of the reduced density matrices that

$$\frac{\partial}{\partial \lambda} \ln Z(\Lambda, ze^{i\lambda}, \beta)\, |_{\lambda=0} = i \int_\Lambda \rho_\Lambda(x, x)\, dx$$

$$\frac{\partial^2}{\partial \lambda^2} \ln Z(\Lambda, ze^{i\lambda}, \beta)\, |_{\lambda=0} = -z \int \frac{\partial}{\partial z} \rho_\Lambda(x, x)\, dx$$

Let $n(\Lambda, z, \beta) = |\Lambda|^{-1} \int_\Lambda \rho_\Lambda(x, x)\, dx$. In Ref. 8 it is shown that

$$zp(\Lambda, z, \beta) = \frac{\partial n}{\partial z} \geqslant \gamma(z, \beta) \tag{4.44}$$

where $\gamma(z, \beta) > 0$, and if $|\Lambda|^{-1} N < c_0^{-1}$; then the equation

$$n(\Lambda, z, \beta) = |\Lambda|^{-1} N$$

has a unique solution $Z(N, \Lambda, \beta)$. Hence, we can find $\varepsilon_\Lambda > 0$, $\delta_{N,\Lambda} > 0$ such that

$$|n(\Lambda, z, \beta) - |\Lambda|^{-1} N| \leqslant |\Lambda|^{-1/2} \tag{4.45}$$

for all $\beta \in (\beta_0 - \varepsilon_\Lambda, \beta_0 + \varepsilon_\Lambda)$, $Z \in [Z(N, \Lambda, \beta_0) - \delta_{N,\lambda}, Z(N, \Lambda, \beta_0) + \delta_{N,\Lambda}]$. We have

$$\int_{|\lambda| < |\Lambda|^{-\alpha}} e^{-iN\lambda} \psi_{\Lambda,z,\beta}(\lambda)\, d\lambda$$

$$= |\Lambda|^{-1/2} \int_{|\mu| < |\Lambda|^{1/2-\alpha}} \exp\{i\mu\, |\Lambda|^{1/2} [n(\Lambda, z, \beta) - |\Lambda|^{-1} N]$$

$$- \frac{\mu^2}{2} zp(\Lambda, z, \beta) + R(\Lambda, z, \beta, |\Lambda|^{-1/2} \mu)\}\, d\mu \tag{4.46}$$

where $\mu = |\Lambda|^{1/2} \lambda$. It follows from (4.25) that

$$\left| \frac{\partial^r}{\partial \beta^r} n(\Lambda, z, \beta) \right| \leqslant b_r(z, \beta), \qquad \left| \frac{\partial^r}{\partial \beta^r} p(\Lambda, z, \beta) \right| \leqslant \tilde{b}_r(z, \beta) \tag{4.47}$$

where b_r, $\tilde{b}_r$ are constants. We can write the remainder R in the form

$$R(\Lambda, z, \beta, \lambda) = \int_0^\lambda dt \int_0^t ds \int_0^s dh \frac{\partial^3}{\partial h^3} \ln Z(\Lambda, ze^{ih}, \beta)$$

Then from (4.25) we have

$$\left| \frac{\partial^r}{\partial \beta^r} R(\Lambda, z, \beta, \lambda) \right| \leqslant \frac{\lambda^3}{6} |\Lambda|\, \tilde{\tilde{b}}_r(z, \beta) \tag{4.48}$$

where $\tilde{\tilde{b}}_r(z, \beta)$ are constants. It follows from (4.38), (4.42), (4.46), (4.45), (4.47), (4.48) that

$$\frac{1}{|\Lambda|} \frac{\partial^r}{\partial \beta^r} \ln Q(N, \Lambda, \beta)\,|_{\beta = \beta_0} = \frac{1}{|\Lambda|} \frac{\partial^r}{\partial \beta^r} \ln Z(\Lambda, z, \beta)\,|_{\beta = \beta_0} + o(1)$$

if $\Lambda \to \infty$, $Z \in [Z(N, \Lambda, \beta_0) - \delta_{N,\lambda}, Z(N, \Lambda, \beta_0) + \delta_{N,\Lambda}]$. Now Theorem 5 follows from Lemma 4.2.

5. PROOF OF THEOREMS 2, 4, AND 6

Let $V(x)$, $\rho_\Lambda(x^m, y^m)$ be the same as in the theorem. In Refs. 1–3 it is shown that the reduced density matrices $\rho_\Lambda(x^m, y^m)$ can be written in the form

$$\rho_\Lambda(x^m, y^m) = \sum_\Pi (\pm)^\Pi \sum_{\mu^m = 0}^{\infty} (\pm)^{|\mu|} \int P^{\mu^m + 1}_{x^m, \Pi(y^m)}(d\omega^m)\, \rho_\Lambda(\omega^m, \mu^m) \tag{5.1}$$

where Λ, x^m, y^m, Π, $(\pm)^{\Pi}$ are the same as in (2.21), $\mu^m = (\mu_1,..., \mu_m)$ is a multi-index, $\mu_i \geqslant 0$ are integers, $|\mu| = \sum_{j=1}^m \mu_j$, $(\pm)^{|\mu|}$, Π is 1 for Bose statistics, $(-1)^{|\mu|}$ for Fermi statistics, $P^{\mu^m+1}_{z^m,w^m}(d\zeta^m)$ denote the conditional Wiener measure on the space

$$\prod_{j=1}^{m} \Omega(z_j, w_j, (\mu_j+1)\beta), \qquad P^{\mu_m+1}_{z^m,w^m}(d\zeta^m) = \prod_j P^{(\mu_j+1)\beta}_{z_j,w_j}(d\zeta_j)$$

if for example $V \in (A)$, then

$$\rho_\Lambda(\omega^m, \mu^m) = \frac{1}{Z(\Lambda, z, \beta)} \sum_{r=0}^{\infty} \frac{z^{m+|\mu|+r}}{r!} \int_{\Lambda^r} du^r \sum_{\Pi} (\pm)^{\Pi} \int P^{\beta}_{u^r,\Pi(u^r)}(d\eta^r)$$
$$\times\, \alpha_{\Lambda,c_0}(\omega^m, \eta^r) \exp[-U(\omega^m, \mu^m, \eta^r)] \tag{5.2}$$

$Z(\Lambda, z, \beta)$ is the grand partition function, $\alpha_{\Lambda,c_0}(\theta^n) = [\prod_{j=1}^n \alpha_\Lambda(\theta_j)]\, \alpha_{\Lambda_{c_0}}(\theta^n)$ where the functional α_Λ is defined by (3.2), $\Lambda_{c_0} = \{(\theta_1,..., \theta_n)$: $\min_{i \neq j} |\theta_i - \theta_j| > c_0\}$

$$U(\omega_1,..., \omega_m, \mu_1,..., \mu_m, \eta_1,..., \eta_r)$$
$$= \int_0^\beta dt \Bigg[\sum_{1 \leqslant i \leqslant j \leqslant m} \sum_{\kappa=0}^{\mu_i} \sum_{l=0}^{\mu_j} V(\omega_i(t+\kappa\beta) - \omega_j(t+l\beta))$$
$$+ \sum_{1 \leqslant i \leqslant m} \sum_{0 \leqslant \kappa < l \leqslant \mu_i} V(\omega_i(t+k\beta) - \omega_i(t+l\beta))$$
$$+ \sum_{1 \leqslant i \leqslant m} \sum_{1 \leqslant j \leqslant r} V(\omega_i(t+k\beta) - \eta_j(t)) \Bigg] \tag{5.3}$$

Let Γ_r denote the set of all equivalence classes of permutations of r variables. We make use of the following notations of Ref. I:

(i) If $\gamma \in \Gamma_r$, then γ_j denotes the number of cycles of length j in the arbitrary permutation $\Pi \in \gamma$, h^r_γ is the number of all permutation $\Pi \in \gamma$, $h^r_\gamma = r!/\prod_{j=1}^r [j^{\gamma_j}(\gamma_j!)]$.

(ii) $h^r_\gamma\, d\gamma = du^r \sum_{\Pi \in \gamma} (\pm)^{\Pi} P_{u^r,\Pi(u^r)}(d\eta^r)$.

(iii) For a given $\Pi \in \gamma$, $\gamma \in \Gamma_r$, and $\eta^r = (\eta_1,..., \eta_r) \in \Omega(u^r, u^r, \beta)$, the trajectories $\eta_1,..., \eta_r$ are distributed in closed loops c corresponding to the cycles $\bar{c}$ of Π; let $\omega_1(s)$ be a trajectory in R^ν, $0 \leqslant s \leqslant (\mu_1+1)_\beta$ then we put

$$K(\omega_1, \mu_1, \gamma) = \sum_{\bar{c} \in \Pi} f(c)$$

where c is the closed loop corresponding to the cycle $\bar{c} \in \Pi$

$$f(c) = \exp\left\{ -\int_0^\beta \sum_{j=0}^{\mu_1} \sum_{\eta_\kappa \in c} V[\omega_1(t+j\beta) - \eta_\kappa(t)]\, dt \right\} - 1$$

(iv) Let ω^m, μ^m be the same as in the Eq. (5.1); then we put

$$F(\omega^m, \mu^m) = \int_0^\beta \Bigg[\sum_{0 \leqslant j < \kappa \leqslant \mu_1} V(\omega_1(t + j\beta) - \omega_1(t + \kappa\beta)) + \sum_{j=0}^{\mu_1} \sum_{\kappa=2}^{m} \sum_{j_\kappa=0}^{\mu_\kappa} V(\omega_1(t + j\beta) - \omega_\kappa(t + j_\kappa\beta)) \Bigg] dt$$

It follows from Ref. 3 that there exists a $z_0 > 0$ such that for $|z| \leqslant z_0$ the 1-point reduced density matrix $\rho_\Lambda(x, y)$ can be written in the following form:

$$\rho_\Lambda(x, y) = \sum_{\mu_1=0}^{\infty} (\pm)^{|\mu_1|} \sum_{q=0}^{\infty} \int P_{x,y}^{(\mu_1+1)\beta}(d\omega_1)\, \zeta_q(\omega_1, \mu_1) \tag{5.4}$$

$$\begin{aligned} \zeta_q(\omega_1, \mu_1) = {} & z^{1+\mu_1+q} \sum_{r_1} \frac{1}{r_1!} \sum_{r_2} \frac{1}{r_2!} \cdots \\ & \times \sum_{r_q} \frac{1}{r_q!} \sum_{\gamma \in \Gamma_{r_1}} \sum_{\delta \in \Gamma_{r_2}} \cdots \sum_{\sigma \in \Gamma_{r_q}} z^{N(\gamma, \delta, \ldots, \sigma)} \\ & \times h_\gamma^{r_1} h_\delta^{r_2} \cdots h_\sigma^{r_q} \int d\gamma \int d\delta \cdots \int d\sigma \\ & \times \exp[-F_1(\omega_1 \mu_1)]\, K(\omega_1, \mu_1, \gamma) \\ & \times \exp[-F(\eta^{(\gamma)}, \varepsilon^{(\gamma)})]\, K(\eta_1^{(\gamma)}, \varepsilon_1^{(\gamma)}, \delta) \\ & \times \exp[-F(\eta^{(\gamma)-1} \varepsilon^{(\gamma)-1}, \theta^{(\delta)} \pi^{(\delta)})] \\ & \times K(\eta_2, \varepsilon_2, \theta^{(\delta)} \pi^{(\delta)}) \cdots \exp[-F(\zeta^{(\lambda)-n+1}, \varkappa^{(\lambda)-n+1}, \ldots) \\ & \times K(\zeta_n^{(\lambda)}, \varkappa_n^{(\lambda)}, \tau^{(\sigma)}, v^{(\sigma)}) \exp[-F(\tau^{(\sigma)}, v^{(\sigma)})]\, \alpha_{\Lambda_{c_0}} \\ & \times \dagger\omega_1(\cdot), \omega_1(\cdot + \beta), \ldots, \omega_1(\cdot + \mu_1\beta), \ldots \cdots) \end{aligned} \tag{5.5}$$

where $\eta^{(\gamma)} = (\eta_1, \eta_2, \ldots, \eta_{\kappa_\gamma})$, $\eta_i(s)$ are the closed loops which correspond to the cycles $\bar{c} \in \Pi$, $\Pi \in \gamma$ [see (ii), (iii)] the trajectory $\eta_i(s)$ is defined for $0 \leqslant s \leqslant (\varepsilon_i + 1)\beta$, $i = 1, 2, \ldots, k_\gamma$, $\varepsilon^\gamma = (\varepsilon_1, \ldots, \varepsilon_{\kappa_\gamma})$; θ^δ, $\pi^\delta, \ldots$ are defined analogously; $\eta^{(\gamma)-1} = (\eta_2, \ldots, \eta_{\kappa_\gamma})$, $\varepsilon^{(\gamma)-1} = (\varepsilon_2, \ldots, \varepsilon_{\kappa_\gamma})$, $\eta^{(\gamma)-2} = (\eta_3, \ldots, \eta_{\kappa_\gamma})$, $\varepsilon^{(\gamma)-2} = (\varepsilon_3, \ldots, \varepsilon_{\kappa_\gamma}) \cdots$; $N(\gamma, \delta, \ldots, \sigma)$ denotes some integer which we shall indicate below; $N(\pi)$ denotes the quantity of all cycles in π, the summation runs over all $\gamma, \delta, \ldots, \sigma$ such that $N(\gamma) + N(\delta) + \cdots + N(\sigma) = q$.

Let us fix $r_1, r_2, \ldots, r_q$, $\gamma \in \Gamma_{r_1}$, $\delta \in \Gamma_{r_2}, \ldots, \sigma \in \Gamma_{r_q}$ in the formula (5.5). Let γ_j be the number of cycles of length j in the permutation $\Pi_1 \in \gamma$, $j = 1, 2, \ldots, r_1$, δ_j be the number of cycles of length j in the permutation $\Pi_2 \in \delta$ and so on. Then the term in the expression $\int P^{\mu_1+1}_{x,y}(d\omega_1)\,\zeta_q(\omega_1,\mu_1)$ which corresponds to the r_1, r_2,..., r_q, γ, δ,..., σ, and arbitrary $\Pi_1 \in \gamma$, $\Pi_2 \in \delta$,... according to (5.5) can be written in the form

$$\begin{aligned}
&\pm \frac{z^{1+\mu_1+q+N(\gamma,\ldots,\sigma)}}{r_1!\,r_2!\cdots r_q!} \iint\cdots\int dx_{11}\,dx_{12}\cdots dx_{1\gamma_1} \\
&\quad\times dx_{21}\,dx_{22}\cdots dx_{2\gamma_2}\cdots dx_{r_1 1\gamma_{r_1}} \\
&\quad\times \iint\cdots\int dy_{11}\cdots dy_{r_2\delta_{r_2}}\cdots \iint\cdots\int dw_{11}\cdots\cdots dw_{r_q\sigma_{r_q}} \\
&\quad\times \iint\cdots\int P^{(\mu_1+1)\beta}_{x,y}(d\omega)\,P^{\beta}_{x_{11},x_{11}}(d\eta_{11})\,P^{\beta}_{x_{12},x_{12}}(d\eta_{12})\cdots P^{\beta}_{x_{1\gamma_1},x_{1\gamma_1}}(d\eta_{1,\gamma_1}) \\
&\quad\times P^{2\beta}_{x_{21},x_{21}}(d\eta_{21})\cdots P^{2\beta}_{x_{2\gamma_2},x_{2\gamma_2}}(d\eta_{2\gamma_2}) \\
&\quad\cdots P^{r_1\beta}_{x_{r_1\gamma_{r_1}},x_{r_1\gamma_{r_1}}}(d\eta_{r_1\gamma_{r_1}})\,P^{\beta}_{y_{11},y_{11}}(d\theta_{11})\cdots P^{r_2\beta}_{y_{r_2\delta_{r_2}},y_{r_2\delta_{r_2}}}(d\eta_{r_2,\delta_{r_2}}) \\
&\quad\cdots P^{\beta}_{w_{11},w_{11}}(d\mu_{11})\cdots\cdots P^{r_q\beta}_{w_{r_q,\sigma_{r_q}},w_{r_q,\sigma_{r_q}}}(d\tau_{r_q,\sigma_{r_q}}) \\
&\quad\times \alpha_\Lambda(\omega)\,\Pi\alpha_\Lambda(\omega_{ij})\,\Pi\alpha_\Lambda(\eta_{ij})\cdots\Pi\alpha_\Lambda(\tau_{ij}) \\
&\quad\times\left(\prod_{i=1}^{\gamma_1}\left\{\exp\left[-\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta)-\eta_{1i}(t))\,dt\right]-1\right\}\right) \\
&\quad\times\left(\prod_{i=1}^{\gamma_2}\left\{\exp\left[-\int_0^\beta \sum_{j=0}^{\mu_1}\sum_{\kappa=0}^{1} V(\omega_1(t+j\beta)-\eta_{2i}(t+\kappa\beta))\,dt\right]-1\right\}\right) \\
&\quad\cdots\cdots\left(\prod_{i=1}^{\delta_1}\left\{\exp\left[-\int_0^\beta V(\eta_{11}(t)-\theta_{1i}(t))\,dt\right]-1\right\}\right) \\
&\quad\cdots\cdots\left(\prod_{i=1}^{\sigma_{r_q}}\left\{\exp\left[-\sum_{\kappa=0}^{\varkappa_{jn}}\right.\right.\right. \\
&\quad\times\left.\left.\left.\sum_{l=0}^{r_q-1} V(-\zeta_{jn}(t+k\beta)-\tau_{r_q,i}(t+l\beta))\,dt\right]-1\right\}\right) \\
&\quad\times\exp[-F(\omega_1,\mu_1)-F(\gamma)-\cdots-F(\zeta_{jn},\varkappa_{jn},\ldots)-F(\tau_{r_q,1},\ldots)]
\end{aligned}$$

$$\begin{aligned}
N(\gamma,\delta,\ldots,\sigma) &= \gamma_2+2\gamma_3+\cdots+(r_1-1)\,\gamma_{r_1} \\
&\quad+\delta_2+\cdots+\lambda_2+2\lambda_3+\cdots+(h-1)\,j
\end{aligned} \tag{5.6}$$

Let us decompose the space R^ν into cubes $\Delta(k_1, k_2,\ldots, k_\nu)$ which have edges with length a_0 where $1 \leqslant \kappa_i < +\infty$, $i=1, 2,\ldots, \nu$. Let θ be a trajectory

$\mathscr{K}_0 = \{(k_1,..., k_\nu)\colon\ \theta \cap \Lambda(k_1,..., k_\nu) \neq \varnothing\}$, $n(\theta) = |\mathscr{K}|$, $\chi_n(\theta) = 1$ if $n(\theta) = n$, $\chi_n(\theta) = 0$ otherwise, $n = 1, 2$. Put

$$\Omega(\omega) = \bigcup_{(\kappa_1,...,\kappa_\nu) \in \mathscr{K}_0} \bigcup_{|\kappa_i - \kappa_i'| \leqslant 1} \Lambda(\kappa_1',..., \kappa_\nu')$$

and for any subset $\Gamma \subset R^\nu$, $[\Gamma]_t = \Gamma$ if $t = 0$, $[\Gamma]_t = R^\nu \backslash \Gamma$ if $t = 1$.

Let us rewrite (5.6) in the following way:

$$\frac{z^{1+\mu+q+N}}{r_1!\, r_2! \cdots r_q!} \sum_{a=1}^{\infty} \sum_{a_{11}=1}^{\infty} \cdots \sum_{a_{1\gamma_1}=1}^{\infty} \cdots\cdots \sum_{a_{r_1\gamma_{r_1}}=1}^{\infty} \sum_{b_{11}=1}^{\infty}$$

$$\cdots\cdots \sum_{b_{r_2\delta_{r_2}}=1}^{\infty} \cdots\cdots \sum_{g_{r_q\sigma_{r_q}}=1}^{\infty} \iint \cdots \int P_{0,y-x}^{(\mu_1+1)\beta}(d\omega)\, P_{0,0}^{\beta}(d\eta_{11})$$

$$\cdots\cdots P_{0,0}^{r_q\beta}(d\tau_{r_q\sigma_{r_q}})\, \chi_a(\omega) \prod \chi_{a_{ij}}(\eta_{ij}) \cdots\cdots \prod \chi_{g_{ij}}(\tau_{ij})$$

$$\times \sum_{t_{111}=0,1} \sum_{t_{112}=0,1} \cdots\cdots \sum_{t_{1r_1\gamma_{r_1}}=0,1} \cdots\cdots \sum_{t_{q,r_q,\tau_{r_q}}=0,1}$$

$$\times \int_{\Omega(x,\omega,\eta_{11},...,\tau_{r_q,\sigma_{r_q}},t_{111},...,t_{q,r_q,\sigma_{r_q}})} \int \cdots\cdots \int dx_{11}\, dx_{12}$$

$$\cdots dw_{r_q\sigma_{r_q}}\, \alpha_{\Lambda,c_0}(\omega + x, \omega_{11} + x_{11}, ... \cdots)$$

$$\times \left(\prod_{i=1}^{\gamma_1} \left\{ \exp\left[-\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta) - \eta_{1i}(t) + x - x_{1i})\, dt \right] - 1 \right\} \right)$$

$$\cdots\cdots \left(\prod_{i=1}^{r_q} \left\{ \exp\left[-\int_0^\beta \sum_{\kappa=0} \sum_{l=0}^{r_q-1} V(\zeta_{jn}(t+\kappa\beta) \right.\right.\right.$$

$$\left.\left.\left. - \tau_{r_q,i}(t+l\beta) + z_{jn} - w_{r_q,i})\, dt \right] - 1 \right\} \right)$$

$$\times \exp[-F(\omega_1 + x, \mu) - \cdots] \tag{5.7}$$

where

$$\Omega(x, \omega, \eta_{11}, \eta_{12}, ... \,..., \tau_{r_q,\sigma_{r_q}}, t_{111}, t_{112}, ... \,..., t_{q,r_q,\sigma_{r_q}})$$

$$= \{x_{ij} - x \in [\Omega(\omega) - \Omega(\eta_{ij})]_{t_{1ij}},\ i = 1, 2,..., r_1,\ j = 1, 2,..., \gamma_i;$$

$$y_{ij} - x_{11} \in [\Omega(\eta_{11}) - \Omega(\theta_{ij})]_{t_{2ij}},\ i = 1, 2,..., r_2,\ j = 1, 2,..., \delta_i, ... \,...;$$

$$w_{li} - z_{jn} \in [\Omega(\zeta_{j,n}) - \Omega(\tau_{li})]_{t_{q,l,i}},\ l = 1, 2,..., r_q,\ i = 1, 2,..., \sigma_l\}$$

Let $\mathscr{T}(a, a_{11}, \dots \dots, g_{r_q,\sigma_{r_q}}, t_{111}, t_{112}, \dots \dots, t_{q,r_q,\tau_{r_q}})$ denote the integral in the expression (5.7), corresponding to the indexes a, $a_{11}, \dots, t_{q,r_q,\tau_{r_q}}$. Let us make the following procedure in the integral $\mathscr{T}$. If $t_{111} = \perp$ then we rewrite the expression

$$\left\{\exp\left[-\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta)-\eta_{11}(t)+x-x_{11})\,dt\right]-1\right\} \tag{5.8}$$

in the form

$$-\int_0^1 d\tilde{t}_{111}\left[\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta)-\eta_{11}(t)+x-x_{11})\right]$$
$$\times\exp\left[-\tilde{t}_{111}\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta)-\eta_{11}(t)+x-x_{11})\,dt\right] \tag{5.9}$$

In the case $t_{111}=0$ we leave the expression (5.8) without change. Analogously we transform the other curly brackets in the integral. Thus we can write

$$\mathscr{T} = \int_0^1 \cdots\cdots \int_0^1 \prod_{ij\in\mathscr{A}_1} d\tilde{t}_{1ij} \prod_{ij\in\tilde{\mathscr{A}}_2} d\tilde{t}_{2ij} \cdots \prod_{ij\in\mathscr{A}_q} d\tilde{t}_{q,ij}$$
$$\times \iint\cdots\int P^{(\mu_1+1)\beta}_{0,y-x}(d\omega)\, P^{\beta}_{0,0}(d\eta_{11})\cdots P^{r_q\beta}_{0,0}(d\tau_{r_q,\sigma_{r_q}})\,\chi_a(\omega)$$
$$\times \prod \chi_{a_{ij}}(\eta_{ij})\cdots\prod \chi_{g_{ig}}(\tau_{ij})$$
$$\times \int_{\Omega(x,\omega,\eta_{11},\dots\dots,\sigma_{q,r_q},t_{111},t_{112},\dots,t_{q,r_q,\sigma_{r_q}})} \int\cdots\cdots\int dx_{11}\,dx_{12}$$
$$\cdots\cdots dw_{r_q,\sigma_{r_q}}\,\alpha_A(\omega+x, \eta_{11}+x_{11},\dots)$$
$$\times \prod_{i\in\mathscr{A}_{11}}\left[-\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t+j\beta)-\eta_{1i}(t)+x-x_{1i})\,dt\right]$$
$$\cdots\cdots \prod_{i\in\mathscr{A}_{q,r_q}}\left[-\int_0^\beta \sum_{\kappa=0}^{\varkappa_{jn}} \sum_{l=0}^{r_q-1} V(\zeta_{jn}(t+\kappa\beta)\right.$$
$$\left.-\tau_{r_q,i}(t+l\beta)+z_{jn}-w_{r_q,i})\,dt\right]$$
$$\times \exp\left[-\int_0^\beta U(\omega(t), \omega(t+\beta),\dots, \omega(t+\mu_1\beta), \eta_{11}(t), \eta_{12}(t),\dots, \eta_{21}(t)\right.$$
$$\left.\eta_{21}(t+\beta), \eta_{22}(t),\dots\dots, \tau_{q,r_q}[t+(r_q-1)\beta])\,dt\right] \tag{5.10}$$

where $\mathscr{A}_\kappa = \{ij: t_{\kappa ij} = \perp\}$, $\mathscr{A}_{\kappa l} = \{i: t_{\kappa li} = 1\}$, $U = U_{t_{111}, t_{112}, \ldots, t_{q, r_q, \sigma_{r_q}}}$ is some continuous function. It follows from the (2.7), that

$$U \geqslant -2(\gamma_1 + 2\gamma_2 + \cdots + r_1 \cdot \gamma_{r_1} + \delta_1 + 2\delta_2 + \cdots \cdots + \lambda_1 + 2\lambda_2 + \cdots + hj)B$$

Let $i \in \mathscr{A}_{q, r_q}$. Then in the expression (5.10) we have

$$|\zeta_{jn}(t + \kappa\beta) - \tau_{r_q, i}(t + l\beta) + z_{ij} - w_{r_q, i}| \geqslant a_0 \tag{5.11}$$

Hence

$$\int dw_{r_q, i} \left[\sum_{\kappa=0}^{h} \sum_{l=0}^{r_q - 1} |V(\zeta_{jn}(t + \kappa\beta) - \tau_{r_q, i}(t + l\beta) + z_{jn} - w_{r_q, i})| \right] \leqslant h \cdot r_q \cdot A \tag{5.12}$$

Furthermore, note that

$$\int_{\Omega(\zeta) - \Omega(\sigma)} dw \leqslant c_1 a_0^2 n(\zeta) \cdot n(\sigma) \tag{5.13}$$

where c_1 is a constant. By using these observations it is possible to show that

$$\begin{aligned} |\mathscr{T}| \leqslant & \left[\sum_{l=1}^{r_1} (\mu_1 l\beta A)^{|\mathscr{A}_{1l}|} \cdot (c_1 a)^{\gamma_l - |\mathscr{A}_{1l}|} \cdot \prod_{i \in A_{1l}} a_{li} \right] \\ & \times \left[\prod_{l=1}^{r_2} (l\beta A)^{|\mathscr{A}_{2l}|} (c_1 a_{11})^{\delta_l - |\mathscr{A}_{2l}|} \cdot \prod_{i \in \mathscr{A}_{2l}} b_{li} \right] \\ & \cdots \cdots \left[\prod_{l=1}^{r_q} (h \cdot l\beta A)^{|\mathscr{A}_{r_q l}|} (c_1 \kappa_{jn})^{\sigma_l - |A_{q,l}|} \prod_{i \in \mathscr{A}_{ql}} g_{li} \right] \\ & \times \exp[2\beta(\gamma_1 + 2\gamma_2 + \cdots + r_i \gamma_{r_1} + \delta_1 + \cdots + h \cdot j)B] \\ & \times P_{0,0}^{\mu_1 \beta}\{\omega: n(\omega) = a\} \prod_{l=1}^{r_1} \prod_{i=1}^{\gamma_l} P_{0,0}^{l\beta}(n(\eta_{li}) = a_{li}) \\ & \cdots \cdots \prod_{l=1}^{r_q} \prod_{i=1}^{\sigma_l} P_{0,0}^{l\beta}(n(\tau_{li}) = g_{l,i}) \end{aligned} \tag{5.14}$$

where c_1 is a constant. It follows from Ref. 3 that for any $t>0$

$$\int P_{0,0}^{l\beta}(d\omega)\exp[tn(\omega)] \leqslant (2\pi l\beta)^{-\nu/2}\, c(t)^l$$

where $c(t) < +\infty$ is a constant. Hence,

$$\begin{aligned} &P_{0,0}^{(\mu_1+1)\beta}(\omega: n(\omega)=a) \\ &\quad\times \prod_{l=1}^{r_1}\prod_{i=1}^{\gamma_l} P_{0,0}^{l\beta}(n(\eta_{li})=a_{li})\cdots \prod_{l=1}^{r_q}\prod_{i=1}^{\sigma_l} P_{0,0}^{l\beta}(n(\tau_{li})=g_{li}) \\ &\leqslant (2\pi\mu_1\beta)^{-\nu/2}(2\pi\beta)^{-\gamma_1\nu/2}\, c(t)^{\mu_1+r_1+\cdots+r_q} \\ &\quad\times \exp\left[-t\left(a+\sum a_{li}+\sum b_{li}+\cdots+\sum g_{li}\right)\right] \end{aligned} \tag{5.15}$$

Now we want to obtain the estimate for the value $\partial\mathscr{T}/\partial\beta$. We shall apply Lemma 3.3. Let $\Lambda(k_1,\ldots,k_\nu)$ be the same as above. Let us denote by $\Gamma_{n,N}$ various connected domains of the type

$$\Gamma_{n,N} = \bigcup_{(\kappa_1,\ldots,\kappa_\nu)\in\mathscr{F}_{n,N}} \Lambda(k_1,\ldots,k_\nu), \qquad |\mathscr{F}_{n,N}| = n,$$

$$0\in\Gamma_{n,N}, \qquad n=1,2,\ldots \tag{5.16}$$

$$P_{up}\chi_{n,N}(\omega)=1 \qquad \text{if} \quad n(\omega)=n, \quad \omega\subset\Gamma_{n,N}$$

$\chi_{n,N}(\omega)=0$ otherwise. A simple argument shows that

$$\chi_n(\omega)=\sum_N \chi_{n,N}(\omega) \tag{5.17}$$

Hence

$$\begin{aligned} &\chi_a(\omega)\prod \chi_{a_{ij}}(\eta_{ij})\cdots\cdots\prod \chi_{g_{ij}}(\tau_{ij}) \\ &\quad= \sum_A \sum_{\mathscr{A}_{ij}}\cdots\sum_{G_{ij}} \chi_{a,\mathscr{A}}(\omega)\prod \chi_{a_{ij},A_{ij}}(\eta_{ij})\prod \chi_{g_{ij}G_{ij}}(\tau_{ij}) \end{aligned} \tag{5.18}$$

We substitute this expression in (5.10) and denote by $\mathscr{T}(A, A_{11},\ldots, G_{r_l\sigma_{r_l}})$ the corresponding term. Let $\chi_{a,A}(\omega)\prod \chi_{a_{ij},A_{ij}}(\eta_{ij})\cdots\prod \chi_{g_{ij}G_{ij}}(\tau_{ij})=1$. Then the domain $\Omega(x,\omega,\eta_{11},\ldots,\tau_{q,\sigma_{r_q}}, t_{111},\ldots, t_{q,r_q,\sigma_{r_q}})$ does not depend on ω, η, $\eta_{11},\ldots,\sigma_{q,r_q}$ and we denote it by $\Omega(x,a,a_{11},\ldots,g_{q,r_q},A,A_{11},\ldots)$. It is evident that a functional of the form of $\chi_{n,N}(\omega)$ can be written as a sum $\chi_{n,N}(\omega)=\sum_{\Gamma\in\mathscr{M}}\alpha_\Gamma(\omega)$ where $|\mathscr{M}|\leqslant 2^n$ and the functionals $\alpha_\Gamma(\omega)$ are defined by (3.2).

Thus we can write the integral $\mathscr{T}(A, A_{11}, \dots\dots, G_{r_l,\sigma_{r_l}})$ as a sum of integrals of the form

$$\mathscr{T}_\varepsilon = \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{ij \in A_1} d\tilde{t}_{1ij} \prod_{i,j \in \mathscr{A}_2} d\tilde{t}_{2ij} \cdots$$

$$\times \int_{\Omega(x,a,a_{11},\dots\dots,G_{r_l,\delta_{r_l}})} \int \cdots\cdots \int dx_{11}\, dx_{12} \cdots\cdots dw_{r_q\sigma_{r_q}}$$

$$\times \iint \cdots \int P^{(\mu_1+1)\beta}_{0,y-x}(d\omega)\, P^{\beta}_{0,0}(d\eta_{11})$$

$$\cdots\cdots P^{r_q\beta}_{0,0}(d\tau_{r_q,\sigma_{r_q}})\, \alpha_\varepsilon(\omega, \eta_{11}, \dots, \tau_{r_q\sigma_{r_q}})$$

$$\times \alpha_A(\omega + x, \eta_{11} + x_{11}, \dots\dots, \tau_{r_q\sigma_{r_q}} + x_{r_q\sigma_{r_q}})$$

$$\times \prod_{i \in \mathscr{A}_{11}} \left[-\int_0^\beta \sum_{j=0}^{\mu_1} V(\omega_1(t + j\beta) - \eta_{1i}(t) + x - x_{1i})\, dt \right]$$

$$\cdots\cdots \prod_{i \in \mathscr{A}_{q,r_q}} \left[-\int_0^\beta \sum_{\kappa=0}^{\varkappa_{jn}} \sum_{l=0}^{r_q-1} V(\zeta_{jn}(t + k\beta) - \tau_{r_q,i}(t + l\beta) \right.$$

$$\left. + z_{jn} - w_{r_q,i})\, dt \right] \exp\left[-\int_0^\beta U(\omega(t), \omega(t+\beta), \dots)\, dt \right] \qquad (5.19)$$

where $\varepsilon \in \mathscr{N}$, $|\mathscr{N}| \leqslant 2^{a + \sum a_{ij} + \cdots + \sum g_{ij}}$ [see expression (5.10)]. Let $\mathscr{L}_\varepsilon$ be the Wiener integral in (5.19). We may estimate the derivatives $\partial^k \mathscr{L}_\varepsilon / \partial \beta^k$ by means of the Lemma. After a simple but enough tedious computations, in which estimates of the type (5.12) are utilized, one can obtain the following estimate:

$$\left| \frac{\partial^2 \mathscr{T}_\varepsilon}{\partial \beta^2} \right| \leqslant [\tilde{c}_2(\beta)]^{\sum l\gamma_l + \sum l\delta_l + \cdots + \sum l\sigma_l} \left[\sum \gamma_l + \sum \delta_l + \cdots + \sum \sigma_l \right]^2$$

$$\times \left[\prod_{l=1}^{r_1} (\mu l \beta A)^{|\mathscr{A}_{1l}|} (c_1 a)^{\gamma_l - |\mathscr{A}_l|} \prod_{i \in \mathscr{A}_{1l}} a_{li} \right]$$

$$\cdots\cdots \left[\prod_{l=1}^{r_q} (\kappa_{jn} l \beta A)^{|\mathscr{A}_{r_q,l}|} (c_1 k_{jn})^{\sigma_l - |\mathscr{A}_{q,l}|} \prod_{i \in \mathscr{A}_{q,l}} g_{li} \right]$$

$$\times \rho(0, y - x, (1 + \mu_1)\beta) \prod_{l=1}^{r_1} (2\pi l \beta)^{-v\gamma_l/2}$$

$$\times \prod_{l=1}^{r_2} (2\pi l \beta)^{-v\delta_l/2} \cdots \prod_{l=1}^{r_q} (2\pi l \beta)^{-v\sigma_l/2} \qquad (5.20)$$

where $\tilde{c}_2(\beta)$ is constant.

It follows from our argument that

$$\left|\frac{\partial^2 \mathscr{T}}{\partial \beta^2}\right| \leqslant c^{a+\sum a_{ij}+\cdots+\sum g_{ij}}\mathscr{G} \tag{5.21}$$

where $\mathscr{G}$ is the quantity on the right-hand side inequality (5.14). Furthermore

$$\left|\frac{\partial \mathscr{T}}{\partial \beta}\right| \leqslant \varepsilon^{-1}\,|\mathscr{T}(\beta+\varepsilon)| + \varepsilon^{-1}\,|\mathscr{T}(\beta)| + \varepsilon \max_{|\beta'-\beta|\leqslant\varepsilon} \left|\frac{\partial^2 \mathscr{T}}{\partial \beta^2}\right|_{\beta=\beta'} \tag{5.22}$$

where $\varepsilon > 0$. Let $t > 2\ln c$, $\varepsilon = \exp[-(t/2)(a+\sum a_{ij}+\cdots+\sum g_{ij}]$. Then from the relations (5.14), (5.21), (5.22) we have

$$\left|\frac{\partial \mathscr{T}}{\partial \beta}\right| \leqslant c(t)^{\mu_1+r_1+\cdots+r_q}\mathscr{G}\exp\left[-\frac{t}{2}\left(a+\sum a_{ij}+\cdots+\sum g_{ij}\right)\right] \tag{5.23}$$

There are $h_\gamma^{r_1}, h_\delta^{r_2}\cdots h_\sigma^{r_q}$ terms of type (5.6) in the expression $\int P_{x,y}^{(\mu_1+1)\beta}(d\omega_1)\,\zeta_q(\omega_1,\mu_1)$. In addition we must multiply the estimates (5.23) by the expression $z^{1+\mu_1+q+N}/(r_1!\cdots r_q!)$ and sum over various $t_{111}, t_{112},\ldots, t_{q,r_q,\sigma_{r_q}}$. It is easy to show that this sum does not exceed the quantity.

$$\begin{aligned} &|z|^{1+\mu_1+q}[c_1(t,\beta)]^q \exp\left\{\left[\mu_1\beta A + q\beta A + c_1\left(a+\sum a_{ij}+\cdots+\sum g_{ij}\right)\right]\right. \\ &\quad\times \left.\sum_{l=1}^{\infty} [|z|\,c_1(t,\beta)]^{l-1}/(2\pi l\beta)^{\nu/2} - \frac{t}{2}\left(a+\sum a_{ij}+\cdots+\sum g_{ij}\right)\right\} \end{aligned} \tag{5.24}$$

where $c_1(t,\beta)$ is a constant. If $2\,|z|\exp(\beta A) < c_1(t,\beta)^{-1}$, $t > 2(2\pi\beta)^{-\nu/2}$ then the sum of these quantities with respect to various μ_1, q, a, $a_{11},\ldots, g_{q,r_q}$ is finite. Thus

$$\left|\frac{\partial}{\partial \beta}\rho_\Lambda(x,y)\right| \leqslant c_1(\beta)$$

where $c_1(\beta)$ is a constant. One can show that if $\beta \in [\beta_1, \beta_2]$, $|z| \leqslant z(\beta_1, \beta_2)$ then

$$\left|\frac{\partial^{r+\kappa}}{\partial \beta^r\,\partial z^\kappa} z^{-m}\rho_\Lambda(x^m, y^m)\right| \leqslant c_{r,\kappa,m} \tag{5.25}$$

where $0 < \beta_1 < \beta_2 < +\infty$, $z(\beta_1, \beta_2) > 0$, $c_{r,\kappa,m}$ are constants $r = 1, 2, 3$, κ, $m = 1, 2,\ldots$. Now the statements of Theorem 6 follow from the results of the Refs. 1–3 and estimates (5.25).

It follows from the relations (5.4), (5.5) and estimates (5.24) that if $z > 0$ is sufficiently small then

$$\frac{\partial^2}{\partial\beta^2}\rho(0,0) > \frac{1}{2}\frac{\partial^2}{\partial\beta^2} z \int P^{\beta}_{x,y}(d\omega) > 0$$

where $\rho(x, y)$ is the limit 1-point reduced density matrix. Thus the hypotheses of Lemma 4.2 are fulfilled (see Section 4), for the grand partition function. Hence Theorem 4 is proved.

The proof of Theorem 2 is similarly to the proof of Theorem 1. In order to take advantage of the method which we have applied in Theorem 1 we must only obtain the estimate for the derivatives $(\partial^r/\partial\beta^r)$ $Z(\Lambda, z, \beta)^{-1} Z(\Lambda, ze^{i\lambda}, \beta)$ where $|\lambda| > |\Lambda|^{-\alpha}$. We have

$$\left|\frac{\partial^r}{\partial\beta^r}\ln Z(\Lambda, \zeta, \beta)\right| \leqslant c_r$$

where, $\beta \in [\beta_1, \beta_2]$, $0 < \beta_1 < \beta_2 < +\infty$, ζ is complex, $|\zeta| < z_1$, c_r are constants, $r = 1, 2, 3$. Hence, if $z > 0$ then

$$\left|\frac{\partial}{\partial\beta} Z(\Lambda, z, \beta)^{-1} Z(\Lambda, ze^{i\lambda}, \beta)\right|$$

$$\leqslant |Z(\Lambda, z, \beta)^{-1} Z(\Lambda, ze^{i\lambda}, \beta)| \left|\frac{\partial}{\partial\beta}\ln Z(\Lambda, z, \beta)^{-1} Z(\Lambda, ze^{i\lambda}, \beta)\right| \leqslant 2c_1$$

Analogously it is possible to estimate the other derivatives Theorem 2 is proved.

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